ECE 3075A
Random Signals

Lecture 9

Characteristic Functions, Normal Distributions

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First and Second Order Moments (cont'd from last lecture)

If we are only to find the first and second order moment of a function of several random variables, we may not need to find the joint density function. Consider $X=\sum_{i=1}^{N} a_i X_i$, a weighted sum of several r.v.s.

$$\overline{X}=E[X]=E\left[\sum_{i=1}^{N} a_i X_i\right]=\sum_{i=1}^{N} a_i E[X_i]=\sum_{i=1}^{N} a_i \overline{X_i}, \qquad X-\overline{X}=\sum_{i=1}^{N} a_i X_i-\sum_{i=1}^{N} a_i \overline{X_i}=\sum_{i=1}^{N} a_i (X_i-\overline{X_i})$$

$$\sigma_X^2=E[(X-\overline{X})^2]=E\left[\sum_{i=1}^{N} a_i (X_i-\overline{X_i})\sum_{j=1}^{N} a_j (X_j-\overline{X_j})\right]$$

$$=\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j E\left[(X_i-\overline{X_i})(X_j-\overline{X_j})\right]=\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j C_{X_i X_j}$$

where $C_{X_i X_j}=E\left[(X_i-\overline{X_i})(X_j-\overline{X_j})\right]$ is the covariance between X_i and X_j

If these r.v.s are uncorrelated, $C_{X_i X_j}=\begin{cases}0, & i\neq j\\ \sigma_{X_i}^2, & i=j\end{cases}$ Then, $\sigma_X^2=\sum_{i=1}^{N} a_i^2 \sigma_{X_i}^2$

The variance of a weighted sum of uncorrelated random variables equals the weighted sum of the variances of the random variables (squared weights).

Characteristic Function

- Definition

$$\Phi_X(u)=E[e^{juX}]=\int_{-\infty}^{\infty} e^{jux} f_X(x)\,dx$$

  that is, the characteristic function of a random variable can be viewed as the Fourier transform of its probability density function.

- The pdf is then the inverse Fourier transform of the characteristic function

$$f_X(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jux}\Phi_X(u)\,du$$

$$\frac{d}{du}\Phi(u)=\int_{-\infty}^{\infty}\left(\frac{d}{du}e^{jux}\right)f_X(x)\,dx=\int_{-\infty}^{\infty} jxe^{jux} f_X(x)\,dx$$

$$\left.\frac{d}{du}\Phi(u)\right|_{u=0}=\int_{-\infty}^{\infty} jx\, f_X(x)\,dx=j\overline{X} \qquad \left.\frac{d^n}{du^n}\Phi(u)\right|_{u=0}=j^n E[X^n]=j^n\overline{X^n}$$

Joint Characteristic Function & Joint Moments

$$\Phi_{X,Y}(u,v)=E\left[e^{j(uX+vY)}\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)e^{j(uX+vY)}dx\,dy$$

$$f_{X,Y}(x,y)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_{X,Y}(u,v)e^{j(uX+vY)}dx\,dy$$

Note that $\Phi_X(u)=\Phi_{X,Y}(u,0)$ and $\Phi_Y(v)=\Phi_{X,Y}(0,v)$. ⟸ Why?

Use characteristic function to find joint moments:

Recall $\left.\frac{d}{du}\Phi(u)\right|_{u=0}=\left.\int_{-\infty}^{\infty}\left(\frac{d}{du}e^{jux}\right)f_X(x)\,dx\right|_{u=0}=\left.\int_{-\infty}^{\infty} jxe^{jux} f_X(x)\,dx\right|_{u=0}=j\overline{X}$

Similarly, $E[XY]=\overline{XY}=-\left[\frac{\partial^2\Phi_{X,Y}(u,v)}{\partial u\partial v}\right]_{u=v=0}$

Joint moments:

$$E[X^nY^k]=\overline{X^nY^k}=\frac{1}{j^{n+k}}\left[\frac{\partial^{n+k}\Phi_{X,Y}(u,v)}{\partial u^n\partial v^k}\right]_{u=v=0}$$

Use Characteristics Function - Example

A random variable X has a probability density function of the form $f_X(x) = 2e^{-2x}u(x)$

Use characteristic function to find the first and second moments of the random variable.

$$\Phi_X(u) = \int_{-\infty}^{\infty} f_X(x)e^{jux}dx = \int_0^{\infty} 2e^{-2x}e^{jux}dx = \left.\frac{2e^{(-2+ju)x}}{-2+ju}\right|_{x=0}^{\infty} = \frac{2}{2-ju}$$

$$\left.\frac{d}{du}\Phi_X(u)\right|_{u=0} = \left.\frac{d}{du}\left(\frac{2}{2-ju}\right)\right|_{u=0} = \left.2j(2-ju)^{-2}\right|_{u=0} = \frac{j}{2} = j\overline{X}$$

$$\left.\frac{d^2}{du^2}\Phi_X(u)\right|_{u=0} = \left.-4(2-ju)^{-3}\right|_{u=0} = -\frac{1}{2} = -\overline{X^2}$$

Thus, $\overline{X} = \frac{1}{2}$ and $\overline{X^2} = \frac{1}{2}$

Relationship betw'n R.V.s through Characteristic Function

Two random variables X and Y have the joint characteristic function

$$\Phi_{X,Y}(u,v) = \exp\left(-2u^2 - 8v^2\right)$$

We can show that they both have zero mean and are uncorrelated.

$$j\overline{X} = \left.\frac{d}{du}\Phi_{X,Y}(u,v)\right|_{u=v=0} = \left.-4u\exp\left(-2u^2-8v^2\right)\right|_{u=v=0} = 0, \quad \therefore \overline{X} = 0$$

$$j\overline{Y} = \left.\frac{d}{dv}\Phi_{X,Y}(u,v)\right|_{u=v=0} = \left.-16v\exp\left(-2u^2-8v^2\right)\right|_{u=v=0} = 0, \quad \therefore \overline{Y} = 0$$

$$(j)^2 R_{XY} = E[XY] = \left.\frac{\partial^2}{\partial u \partial v}\Phi_{X,Y}(u,v)\right|_{u=v=0} = \left.\frac{\partial^2}{\partial u \partial v}\exp\left(-2u^2-8v^2\right)\right|_{u=v=0}$$

$$= \left.(-4u)(-16v)\exp\left(-2u^2-8v^2\right)\right|_{u=v=0} = 0, \quad \therefore R_{XY} = 0 = \overline{X}\overline{Y},$$

Therefore, X and Y are uncorrelated.

Characteristic Function of Sum of R.V.s

- The characteristic function of a random variable is the Fourier transform of its pdf.
- Very useful in deriving the moments of the r.v.

$$\Phi_X(u) = \int_{-\infty}^{\infty} f_X(x)e^{jux}dx \quad \text{and} \quad f_X(x) = (2\pi)^{-1}\int_{-\infty}^{\infty}\Phi_X(u)e^{-jux}du$$

$$\Phi_Y(u) = \int_{-\infty}^{\infty} f_Y(y)e^{juy}dy \quad \text{and} \quad f_Y(y) = (2\pi)^{-1}\int_{-\infty}^{\infty}\Phi_Y(u)e^{-juy}du$$

- Now, $Z = X + Y$ where X and Y are independent,

The pdf of the sum of two statistically independent random variables is the convolution of their individual pdf's.

$$\Phi_Z(u) = \Phi_X(u)\Phi_Y(u)$$

$$f_Z(z) = (2\pi)^{-1}\int_{-\infty}^{\infty}\Phi_Z(u)e^{-juz}du = (2\pi)^{-1}\int_{-\infty}^{\infty}\Phi_X(u)\Phi_Y(u)e^{-juz}du$$

Characteristic Function of Sum of R.V.s

- Extend the previous result to sum of many r.v.s

Let $Y = X_1 + X_2 + \cdots X_N$ where $X_1, X_2, \cdots X_N$ are statistically independent r.v.s, with pdf and characteristic functions, $f_{X_i}(x_i)$ and $\Phi_{X_i}(u_i)$, respectively. Their joint characteristic function is thus

$$\Phi_{X_1,X_2,\cdots,X_N}(u_1,u_2,\cdots,u_N) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{X_1,X_2,\cdots,X_N}(x_1,x_2,\cdots,x_N)\exp\left[j\sum_{i=1}^{N}u_i x_i\right]dx_1 dx_2\cdots dx_N$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\prod_{i=1}^{N} f_{X_i}(x_i)\right]\exp\left[j\sum_{i=1}^{N}u_i x_i\right]dx_1 dx_2\cdots dx_N = \prod_{i=1}^{N}\int_{-\infty}^{\infty} f_{X_i}(x_i)e^{ju_i x_i}dx_i = \prod_{i=1}^{N}\Phi_{X_i}(u_i)$$

Now, $\Phi_Y(v) = E[e^{jvY}] = E\left[\exp\left(j\sum_{i=1}^{N}vX_i\right)\right] = \Phi_{X_1,X_2,\cdots,X_N}(v,v,\cdots,v) = \prod_{i=1}^{N}\Phi_{X_i}(v)$

and $f_Y(y) = (2\pi)^{-1}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\Phi_{X_i}(v)e^{-jvy}dv$

$\Phi_Y(v)$ is equal to the joint characteristic function along the line $u_1 = u_2 = \cdots = u_N = v$

$f_Y(y) = (2\pi)^{-1}\int_{-\infty}^{\infty}\left[\Phi_X(v)\right]^N e^{-jvy}dv$ if X_i are i.i.d. and $\Phi_{X_i}(v) = \Phi_X(v)$

Density Function of Sum of Several R.V.s

$Y = X_1 + X_2 + \cdots X_N$ where $X_1, X_2, \cdots X_N$ are statistically independent r.v.s, with pdf $f_{X_i}(x_i)$.

$f_Y(y)$ can be obtained by $(N\text{-}1)$-fold convolution of the N individual pdfs.

Let $Y_1 = X_1 + X_2$. Then, $f_{Y_1}(y_1) = f_{X_1}(x_1) * f_{X_2}(x_2)$

Let $Y_2 = Y_1 + X_3$. Then, $f_{Y_2}(y_2) = f_{Y_1}(y_1) * f_{X_3}(x_3)$

$$= f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3)$$

because Y_1 and X_3 are independent.

The same applies to the rest of the r.v.s and thus

$$f_Y(y) = f_{X_1}(x_1) * f_{X_2}(x_2) * \cdots * f_{X_{N-1}}(x_{N-1}) * f_{X_N}(x_N)$$

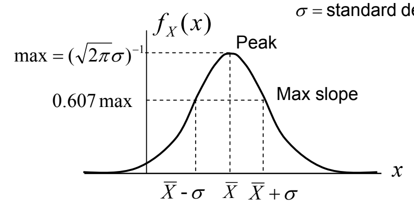
Gaussian Distributions

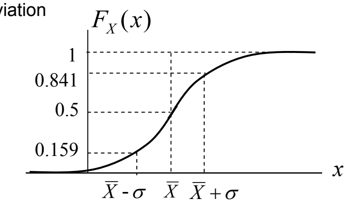
- A r.v. X is gaussian if its pdf is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\overline{X})^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

where $\overline{X}$ and σ^2 are called the mean and variance, respectively.

σ = standard deviation





- ✓ Also called normal distribution, denoted as $\mathcal{N}(x; \overline{X}, \sigma^2)$
- ✓ Pdf has a single peak.
- ✓ $\delta(x-\overline{X}) = \lim_{\sigma\to 0}(\sqrt{2\pi}\sigma)^{-1}\exp\left[-(x-\overline{X})^2/(2\sigma^2)\right]$, a good representation for a delta function because a Gaussian pdf is infinitely differentiable.

Characteristic Function of A Gaussian R.V.

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{jux} f_X(x)\,dx$$

$$\Phi_X(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(jux) \exp\left[\frac{-(x-\overline{X})^2}{2\sigma^2}\right] dx$$

The exponential term:

$$jux - \frac{(x-\overline{X})^2}{2\sigma^2} = \frac{jux\,2\sigma^2 - (x-\overline{X})^2}{2\sigma^2} = -\frac{\left[x-(\overline{X}+ju\sigma^2)\right]^2}{2\sigma^2} + ju\overline{X} - \frac{u^2\sigma^2}{2}$$

$$\Phi_X(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left[x-(\overline{X}+ju\sigma^2)\right]^2}{2\sigma^2} + ju\overline{X} - \frac{u^2\sigma^2}{2}\right] dx$$

$$= \left\{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left[x-(\overline{X}+ju\sigma^2)\right]^2}{2\sigma^2}\right] dx\right\} \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right) = \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)$$

because $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\left[x-(\overline{X}+ju\sigma^2)\right]^2}{2\sigma^2}\right] dx = 1$

Use Cauchy theorem for contour integral – net result is that it is just the same as integrating along the real axis.

Moments of Gaussian Random Variable

$$\frac{d^n}{du^n}\Phi(u)\Big|_{u=0} = j^n E[X^n] = j^n \overline{X^n}$$

$$\Phi_X(u) = \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)$$

$$\frac{d}{du}\Phi(u)\Big|_{u=0} = (j\overline{X} - u\sigma^2)\exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)\Big|_{u=0} = j\overline{X}$$

$$\frac{d^2}{du^2}\Phi(u)\Big|_{u=0} = -\sigma^2\exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right) + (j\overline{X} - u\sigma^2)^2\exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)\Big|_{u=0}$$

$$= -\sigma^2 - \overline{X}^2 = -\overline{X^2} = j^2\overline{X^2}$$

$$\overline{X^2} = \sigma^2 + \overline{X}^2$$

$$\frac{d^3}{du^3}\Phi(u)\Big|_{u=0} = \left(-3\sigma^2(j\overline{X} - u\sigma^2) + (j\overline{X} - u\sigma^2)^3\right)\exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)\Big|_{u=0}$$

$$= -3\sigma^2 j\overline{X} - j\overline{X}^3 = -j(3\sigma^2\overline{X} + \overline{X}^3) = j^3\overline{X^3}$$

$$\overline{X^3} = 3\sigma^2\overline{X} + \overline{X}^3$$

$$\frac{d^4}{du^4}\Phi(u)\Big|_{u=0} = \left(3\sigma^4 - 6\sigma^2(j\overline{X} - u\sigma^2)^2 + (j\overline{X} - u\sigma^2)^4\right)\exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)\Big|_{u=0}$$

$$= 3\sigma^4 + 6\sigma^2\overline{X}^2 + \overline{X}^4 = j^4\overline{X^4} = \overline{X^4}$$