

ECE 3075A
Random Signals

Lecture 9

Characteristic Functions, Normal Distributions

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First and Second Order Moments (cont'd from last lecture)

If we are only to find the first and second order moment of a function of several random variables, we may not need to find the joint density function. Consider $X = \sum_{i=1}^N a_i X_i$, a weighted sum of several r.v.s.

$$\bar{X} = E[X] = E\left[\sum_{i=1}^N a_i X_i\right] = \sum_{i=1}^N a_i E[X_i] = \sum_{i=1}^N a_i \bar{X}_i, \quad X - \bar{X} = \sum_{i=1}^N a_i X_i - \sum_{i=1}^N a_i \bar{X}_i = \sum_{i=1}^N a_i (X_i - \bar{X}_i)$$

$$\sigma_X^2 = E[(X - \bar{X})^2] = E\left[\sum_{i=1}^N a_i (X_i - \bar{X}_i) \sum_{j=1}^N a_j (X_j - \bar{X}_j)\right]$$

$$= \sum_{i=1}^N \sum_{j=1}^N a_i a_j E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)] = \sum_{i=1}^N \sum_{j=1}^N a_i a_j C_{X_i X_j}$$

where $C_{X_i X_j} = E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)]$ is the covariance between X_i and X_j

If these r.v.s are uncorrelated, $C_{X_i X_j} = \begin{cases} 0, & i \neq j \\ \sigma_{X_i}^2, & i = j \end{cases}$ Then, $\sigma_X^2 = \sum_{i=1}^N a_i^2 \sigma_{X_i}^2$

The variance of a weighted sum of uncorrelated random variables equals the weighted sum of the variances of the random variables (squared weights).

Characteristic Function

- Definition

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{jux} f_X(x) dx$$

that is, the characteristic function of a random variable can be viewed as the Fourier transform of its probability density function.

- The pdf is then the inverse Fourier transform of the characteristic function

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jux} \Phi_X(u) du$$

$$\frac{d}{du} \Phi(u) = \int_{-\infty}^{\infty} \left(\frac{d}{du} e^{jux} \right) f_X(x) dx = \int_{-\infty}^{\infty} jx e^{jux} f_X(x) dx$$

$$\left. \frac{d}{du} \Phi(u) \right|_{u=0} = \int_{-\infty}^{\infty} jx f_X(x) dx = j\bar{X}$$

$$\left. \frac{d^n}{du^n} \Phi(u) \right|_{u=0} = j^n E[X^n] = j^n \overline{X^n}$$

Joint Characteristic Function & Joint Moments

$$\Phi_{X,Y}(u, v) = E[e^{j(uX+vY)}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) e^{j(uX+vY)} dx dy$$

$$f_{X,Y}(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{X,Y}(u, v) e^{j(uX+vY)} dx dy$$

Note that $\Phi_X(u) = \Phi_{X,Y}(u, 0)$ and $\Phi_Y(v) = \Phi_{X,Y}(0, v)$. \leftarrow Why?

Use characteristic function to find joint moments:

$$\text{Recall } \frac{d}{du} \Phi(u) \Big|_{u=0} = \int_{-\infty}^{\infty} \left(\frac{d}{du} e^{jux} \right) f_X(x) dx \Big|_{u=0} = \int_{-\infty}^{\infty} jx e^{jux} f_X(x) dx \Big|_{u=0} = j\bar{X}$$

$$\text{Similarly, } E[XY] = \overline{XY} = - \left[\frac{\partial^2 \Phi_{X,Y}(u, v)}{\partial u \partial v} \right]_{u=v=0}$$

Joint moments:

$$E[X^n Y^k] = \overline{X^n Y^k} = \frac{1}{j^{n+k}} \left[\frac{\partial^{n+k} \Phi_{X,Y}(u, v)}{\partial u^n \partial v^k} \right]_{u=v=0}$$

Use Characteristics Function - Example

A random variable X has a probability density function of the form

$$f_X(x) = 2e^{-2x}u(x)$$

Use characteristic function to find the first and second moments of the random variable.

$$\Phi_X(u) = \int_{-\infty}^{\infty} f_X(x)e^{jux} dx = \int_0^{\infty} 2e^{-2x}e^{jux} dx = \frac{2e^{(-2+ju)x}}{-2+ju} \Big|_{x=0}^{\infty} = \frac{2}{2-ju}$$

$$\frac{d}{du} \Phi_X(u) \Big|_{u=0} = \frac{d}{du} \left(\frac{2}{2-ju} \right) \Big|_{u=0} = 2j(2-ju)^{-2} \Big|_{u=0} = \frac{j}{2} = j\bar{X}$$

$$\frac{d^2}{du^2} \Phi_X(u) \Big|_{u=0} = -4(2-ju)^{-3} \Big|_{u=0} = -\frac{1}{2} = -\overline{X^2} \quad \text{Thus, } \bar{X} = \frac{1}{2} \text{ and } \overline{X^2} = \frac{1}{2}$$

Relationship betw'n R.V.s through Characteristic Function

Two random variables X and Y have the joint characteristic function

$$\Phi_{X,Y}(u, v) = \exp(-2u^2 - 8v^2)$$

We can show that they both have zero mean and are uncorrelated.

$$j\bar{X} = \frac{d}{du} \Phi_{X,Y}(u, v) \Big|_{u=v=0} = -4u \exp(-2u^2 - 8v^2) \Big|_{u=v=0} = 0, \quad \therefore \bar{X} = 0$$

$$j\bar{Y} = \frac{d}{dv} \Phi_{X,Y}(u, v) \Big|_{u=v=0} = -16v \exp(-2u^2 - 8v^2) \Big|_{u=v=0} = 0, \quad \therefore \bar{Y} = 0$$

$$\begin{aligned} (j)^2 R_{XY} &= E[XY] = \frac{\partial^2}{\partial u \partial v} \Phi_{X,Y}(u, v) \Big|_{u=v=0} = \frac{\partial^2}{\partial u \partial v} \exp(-2u^2 - 8v^2) \Big|_{u=v=0} \\ &= (-4u)(-16v) \exp(-2u^2 - 8v^2) \Big|_{u=v=0} = 0, \quad \therefore R_{XY} = 0 = \bar{X}\bar{Y}, \end{aligned}$$

Therefore, X and Y are uncorrelated.

Characteristic Function of Sum of R.V.s

- The characteristic function of a random variable is the Fourier transform of its pdf.
- Very useful in deriving the moments of the r.v.

$$\Phi_X(u) = \int_{-\infty}^{\infty} f_X(x)e^{jux} dx \quad \text{and} \quad f_X(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \Phi_X(u)e^{-jux} du$$

$$\Phi_Y(u) = \int_{-\infty}^{\infty} f_Y(y)e^{juy} dy \quad \text{and} \quad f_Y(y) = (2\pi)^{-1} \int_{-\infty}^{\infty} \Phi_Y(u)e^{-juy} du$$

- Now, $Z = X + Y$ where X and Y are independent,

The pdf of the sum of two statistically independent random variables is the convolution of their individual pdf's.

$$\Phi_Z(u) = \Phi_X(u)\Phi_Y(u)$$

$$f_Z(z) = (2\pi)^{-1} \int_{-\infty}^{\infty} \Phi_Z(u)e^{-juz} du = (2\pi)^{-1} \int_{-\infty}^{\infty} \Phi_X(u)\Phi_Y(u)e^{-juz} du$$

Characteristic Function of Sum of R.V.s

- Extend the previous result to sum of many r.v.s

Let $Y = X_1 + X_2 + \dots + X_N$ where X_1, X_2, \dots, X_N are statistically independent r.v.s, with pdf and characteristic functions, $f_{X_i}(x_i)$ and $\Phi_{X_i}(u_i)$, respectively. Their joint characteristic function is thus

$$\begin{aligned} \Phi_{X_1, X_2, \dots, X_N}(u_1, u_2, \dots, u_N) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_{X_1, X_2, \dots, X_N}(x_1, x_2, \dots, x_N) \exp\left[j \sum_{i=1}^N u_i x_i\right] dx_1 dx_2 \dots dx_N \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left[\prod_{i=1}^N f_{X_i}(x_i) \right] \exp\left[j \sum_{i=1}^N u_i x_i\right] dx_1 dx_2 \dots dx_N = \prod_{i=1}^N \int_{-\infty}^{\infty} f_{X_i}(x_i) e^{ju_i x_i} dx_i = \prod_{i=1}^N \Phi_{X_i}(u_i) \end{aligned}$$

Now,
$$\Phi_Y(v) = E[e^{jvY}] = E\left[\exp\left(j \sum_{i=1}^N v X_i\right)\right] = \Phi_{X_1, X_2, \dots, X_N}(v, v, \dots, v) = \prod_{i=1}^N \Phi_{X_i}(v)$$

and
$$f_Y(y) = (2\pi)^{-1} \int_{-\infty}^{\infty} \prod_{i=1}^N \Phi_{X_i}(v) e^{-jvy} dv$$

$\Phi_Y(v)$ is equal to the joint characteristic function along the line $u_1 = u_2 = \dots = u_N = v$

$$f_Y(y) = (2\pi)^{-1} \int_{-\infty}^{\infty} [\Phi_X(v)]^N e^{-jvy} dv \quad \text{if } X_i \text{ are i.i.d. and } \Phi_{X_i}(v) = \Phi_X(v)$$

Density Function of Sum of Several R.V.s

$Y = X_1 + X_2 + \cdots + X_N$ where X_1, X_2, \cdots, X_N are statistically independent r.v.s, with pdf $f_{X_i}(x_i)$.

$f_Y(y)$ can be obtained by $(N-1)$ -fold convolution of the N individual pdfs.

Let $Y_1 = X_1 + X_2$. Then, $f_{Y_1}(y_1) = f_{X_1}(x_1) * f_{X_2}(x_2)$

Let $Y_2 = Y_1 + X_3$. Then, $f_{Y_2}(y_2) = f_{Y_1}(y_1) * f_{X_3}(x_3)$
 $= f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3)$

because Y_1 and X_3 are independent.

The same applies to the rest of the r.v.s and thus

$$f_Y(y) = f_{X_1}(x_1) * f_{X_2}(x_2) * \cdots * f_{X_{N-1}}(x_{N-1}) * f_{X_N}(x_N)$$

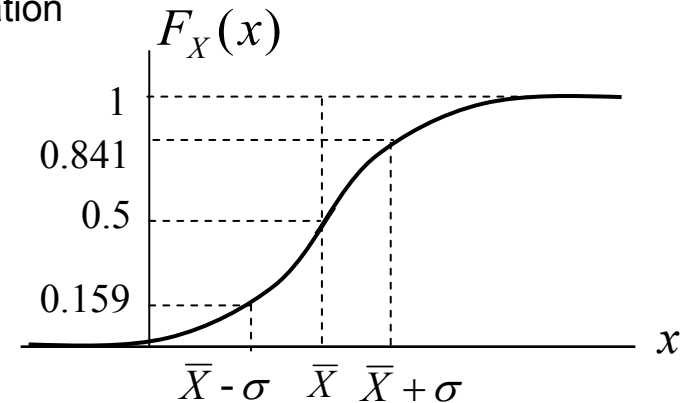
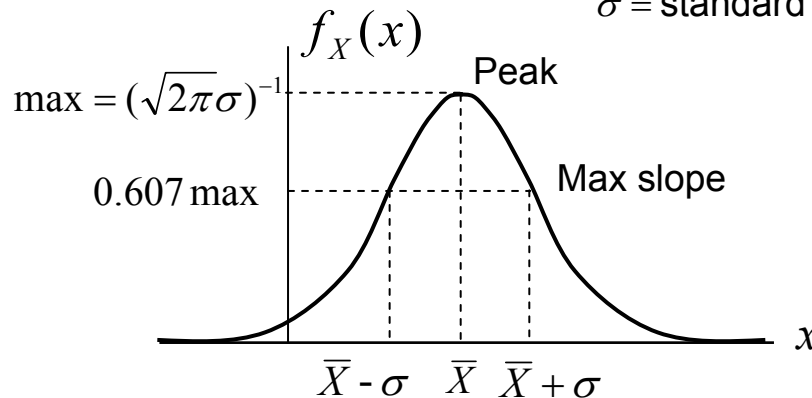
Gaussian Distributions

- A r.v. X is gaussian if its pdf is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x - \bar{X})^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

where \bar{X} and σ^2 are called the mean and variance, respectively.

σ = standard deviation



- ✓ Also called normal distribution, denoted as $\mathcal{N}(x; \bar{X}, \sigma^2)$
- ✓ Pdf has a single peak.
- ✓ $\delta(x - \bar{X}) = \lim_{\sigma \rightarrow 0} (\sqrt{2\pi}\sigma)^{-1} \exp[-(x - \bar{X})^2 / (2\sigma^2)]$, a good representation for a delta function because a Gaussian pdf is infinitely differentiable.

Characteristic Function of A Gaussian R.V.

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{jux} f_X(x) dx$$

$$\Phi_X(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(jux) \exp\left[-\frac{(x-\bar{X})^2}{2\sigma^2}\right] dx$$

The exponential term:

$$jux - \frac{(x-\bar{X})^2}{2\sigma^2} = \frac{jux 2\sigma^2 - (x-\bar{X})^2}{2\sigma^2} = -\frac{[x - (\bar{X} + ju\sigma^2)]^2}{2\sigma^2} + ju\bar{X} - \frac{u^2\sigma^2}{2}$$

$$\begin{aligned} \Phi_X(u) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{[x - (\bar{X} + ju\sigma^2)]^2}{2\sigma^2} + ju\bar{X} - \frac{u^2\sigma^2}{2}\right] dx \\ &= \left\{ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{[x - (\bar{X} + ju\sigma^2)]^2}{2\sigma^2}\right] dx \right\} \exp\left(ju\bar{X} - \frac{u^2\sigma^2}{2}\right) = \exp\left(ju\bar{X} - \frac{u^2\sigma^2}{2}\right) \end{aligned}$$

because
$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{[x - (\bar{X} + ju\sigma^2)]^2}{2\sigma^2}\right] dx = 1$$

Use Cauchy theorem for contour integral – net result is that it is just the same as integrating along the real axis.

Moments of Gaussian Random Variable

$$\frac{d^n}{du^n} \Phi(u) \Big|_{u=0} = j^n E[X^n] = j^n \overline{X^n}$$

$$\Phi_X(u) = \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right)$$

$$\frac{d}{du} \Phi(u) \Big|_{u=0} = (j\overline{X} - u\sigma^2) \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right) \Big|_{u=0} = j\overline{X}$$

$$\begin{aligned} \frac{d^2}{du^2} \Phi(u) \Big|_{u=0} &= -\sigma^2 \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right) + (j\overline{X} - u\sigma^2)^2 \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right) \Big|_{u=0} \\ &= -\sigma^2 - \overline{X}^2 = -\overline{X^2} = j^2 \overline{X^2} \end{aligned}$$

$$\overline{X^2} = \sigma^2 + \overline{X}^2$$

$$\frac{d^3}{du^3} \Phi(u) \Big|_{u=0} = \left(-3\sigma^2(j\overline{X} - u\sigma^2) + (j\overline{X} - u\sigma^2)^3\right) \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right) \Big|_{u=0}$$

$$= -3\sigma^2 j\overline{X} - j\overline{X}^3 = -j(3\sigma^2 \overline{X} + \overline{X}^3) = j^3 \overline{X^3}$$

$$\overline{X^3} = 3\sigma^2 \overline{X} + \overline{X}^3$$

$$\frac{d^4}{du^4} \Phi(u) \Big|_{u=0} = \left(3\sigma^4 - 6\sigma^2(j\overline{X} - u\sigma^2)^2 + (j\overline{X} - u\sigma^2)^4\right) \exp\left(ju\overline{X} - \frac{u^2\sigma^2}{2}\right) \Big|_{u=0}$$

$$= 3\sigma^4 + 6\sigma^2 \overline{X}^2 + \overline{X}^4 = j^4 \overline{X^4} = \overline{X^4}$$