

# ECE 3075A Random Signals

## Lecture 24 Final Review

School of Electrical and Computer Engineering  
Georgia Institute of Technology  
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# Approaches to the Theory of Probability

- Relative frequency approach
  - An event that occurs more frequently has higher probability and vice versa; difficult to generalize
- Axiomatic approach
  - Probability is a real number between 0 and 1;
  - Built upon set theory;
  - A **Probability Space** consists of three components: the observation space,  $S$ , the probability measure,  $\Pr()$ , and the assignment of probability,  $\Pr(\text{event})$ , that satisfies a set of axioms.
  - **Observation or Sample Space** is a space whose elements are all the outcomes of an experiment.
  - **Events** are subsets of the observation space.
  - **Three axioms** to satisfy:

Non - negativity :  $\Pr(A) \geq 0$

Sure event & total probability :  $\Pr(S) = 1$

Exclusivity : If  $A \cap B = \phi$ , then  $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

## Important Corollaries

- ❖ 0-measure events:

$$\Pr(\phi) = 0$$

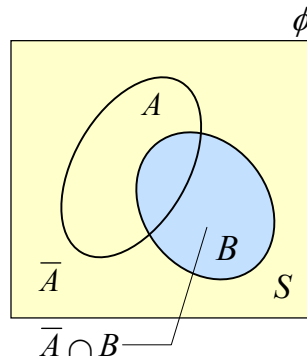
Theoretically, there could be other 0-measure events than the empty set – the axioms allow that. But in the relative frequency approach, they are usually either not treated or treated differently as “unseen events”.

- ❖ Complementary events:

$$\Pr(A) = 1 - \Pr(\bar{A}) \leq 1$$

- ❖  $\Pr(A \cup B) = \Pr(A) + \Pr(\bar{A} \cap B)$

- ❖  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \leq \Pr(A) + \Pr(B)$

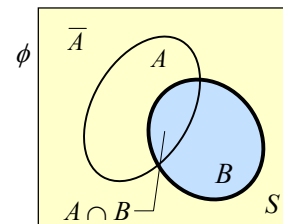


## Conditional Probability

- If event  $B$  is assumed to have non-zero probability, the conditional probability of  $A$ , given  $B$ , is

$$\Pr(A|B) = \frac{\Pr(A, B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)}, \quad \Pr(B) > 0$$

- We say joint event  $(A, B)$  occurs if the outcome of a trial satisfies the definition of both events, in other words, when any of the common elements of  $A$  and  $B$  appears as the outcome of the trial. That is,  $\Pr(A, B) = \Pr(A \cap B)$



$$\Pr(B|B) = 1$$

$\Pr(A|B)$  can be considered as the probability of  $A$  when  $B$  is the observation space.

$$(A \cap B) \cup (\bar{A} \cap B) = S \cap B = B,$$

$$\text{and } (A \cap B) \cap (\bar{A} \cap B) = \phi$$

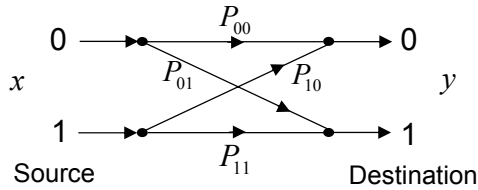
$$\Rightarrow P(A, B) + P(\bar{A}, B) = P(B)$$

# Probability & Information Transmission

## A binary channel

At the source,  
 $\Pr(x=0) = p$  and  
 $\Pr(x=1) = 1-p$ ,

$\Pr(x=0)$  and  $\Pr(x=1)$   
 are called **a priori** probabilities.



The channel, being non-ideal (causing confusions), is characterized by the four conditional probabilities:  $P_{00}$ ,  $P_{01}$ ,  $P_{10}$ , and  $P_{11}$

$P_{ij} = \Pr(y = j | x = i)$  (probability of  $j$  received at the destination, i.e.,  $y = j$ , when  $i$  was actually sent by the source, i.e.,  $x = i$ )

### A posteriori probability –

The probability that  $i$  was sent at the source, given that  $j$  is received at the destination.

$$\Pr(x = i | y = j) = \frac{\Pr(y = j | x = i) \Pr(x = i)}{\Pr(y = j)}$$

# Bayes Formula

Is used to find a posteriori probability

$$\Pr(x = i | y = j) = \frac{\Pr(y = j | x = i) \Pr(x = i)}{\Pr(y = j)}$$

More precisely,

$\Pr(x = i | y = j) = \Pr(\text{symbol } i \text{ was sent, given that } j \text{ is received})$

$$= \frac{\Pr(y = j | x = i) \Pr(x = i)}{\sum_{\text{all } i} \Pr(y = j | x = i) \Pr(x = i)}$$

$$= \frac{P_{ij} \Pr(x = i)}{\sum_{\text{all } i} P_{ij} \Pr(x = i)} = \frac{P_{ij} \Pr(x = i)}{\Pr(y = j)} = \text{the source portion of contributions that led to the reception of } j$$

# Probability Distribution Function

- $\Pr(X \leq x)$  is a **function** of  $x$ .

$F_X(x) = \Pr(X \leq x)$  is the **probability distribution function** defined over all  $x$ .

$$\{X \leq -\infty\} = \phi, \quad F_X(-\infty) = 0$$

$\{X \leq \infty\}$  is always true, a sure event, thus,  $F_X(\infty) = 1$

$$\{X \leq -\infty\} \subset \{X \leq x\} \subset \{X \leq \infty\} \quad \text{for } -\infty < x < \infty$$

$$\Rightarrow 0 \leq F_X(x) \leq 1$$

- If  $x_1 < x_2$ ,  $\{X \leq x_1\} \subset \{X \leq x_2\}$ ,  $\{X \leq x_1\} \cap \{X \leq x_2\} = \{X \leq x_1\}$   
 $\{X \leq x_2\} = \{X \leq x_1\} \cup \{x_1 < X \leq x_2\}$ , and  $\{X \leq x_1\} \cap \{x_1 < X \leq x_2\} = \phi$

$$\therefore \Pr(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$$

$F_X(x)$  is a non-decreasing function of  $x$ .

# Probability Density Functions

- The slope of the probability distribution function at  $x$  represents the incremental probability at that point and thus gives the sense of how likely  $X = x$  might be.

$$f_X(x) = \lim_{\epsilon \rightarrow 0} \frac{F_X(x + \epsilon) - F_X(x)}{\epsilon} = \frac{dF_X(x)}{dx}$$

$$f_X(x) dx = \Pr(x < X \leq x + dx)$$

is the probability mass at  $x$ .

- The derivative is called the probability density function (pdf). Pdf is **non-negative**. In the case of discrete distributions, the pdf consists of delta functions at those realizable values, each having an area equal to the corresponding magnitude of probability.

## Functions of Random Variable

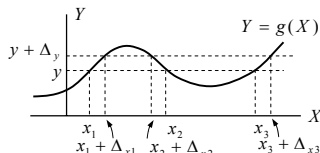
- $X$  is a random variable with pdf  $f_X(x)$ .
- $Y$  is a monotonic function of  $X$ ;  $Y = g(X)$ . Find  $f_Y(y)$ .

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Expressed in  $y$  with  $g^{-1}(y) = x$ ,  $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$

- If  $Y$  is non-monotonic function of  $X$ ,

$$f_Y(y) = \sum_{\text{for all } x=g^{-1}(y)} f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$



Example:  $Y = X^2$  or  $X = \sqrt{Y}$ ,  $\left| \frac{dx}{dy} \right| = \frac{1}{2\sqrt{y}}$ , but for any  $y > 0$ ,  $x = \pm\sqrt{y}$

Therefore,  $f_Y(y) = \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})]$ ,  $y > 0$ ;  $= 0$ ,  $y < 0$

## Mean Values and Moments

- Mean or expected value of a random variable

$$\bar{X} = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Mean or expected value of a function of a random variable

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

- $N^{\text{th}}$  moments & central moments of a r.v.

$$\overline{X^n} = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx \quad \overline{(X - \bar{X})^n} = E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{X})^n f_X(x) dx$$

When  $n = 2$ ,

$\overline{X^2} = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$  is called the mean-square value

and  $\sigma^2 = E[(X - \bar{X})^2] = \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x) dx$  is the variance.

$$\sigma^2 = E[(X - \bar{X})^2] = E[X^2 - 2X\bar{X} + \bar{X}^2] = E[X^2] - 2\bar{X}^2 + \bar{X}^2 = E[X^2] - \bar{X}^2$$

## Conditional Probability Distribution

We define the conditional probability the same as before.

$$F_X(x | M) = \Pr(X \leq x | M) = \frac{\Pr(X \leq x, M)}{\Pr(M)}, \quad \Pr(M) > 0$$

If we use the event mapping concept,  $\Pr(X \leq x, M)$  is the probability of all the outcomes which realize both events  $X(\xi) \leq x$  and  $\xi \in M$ .

$$\text{If } M = \{X \leq m\}, \quad F_X(x | M) = \Pr(X \leq x | X \leq m) = \frac{\Pr(X \leq x, X \leq m)}{\Pr(X \leq m)}$$

$$\text{If } x \leq m, \quad F_X(x | M) = \frac{\Pr(X \leq x)}{\Pr(X \leq m)} = \frac{F_X(x)}{F_X(m)} \quad \text{If } x \geq m, \quad F_X(x | M) = \frac{\Pr(X \leq m)}{\Pr(X \leq m)} = 1$$

Conditional probability density function has all the properties of a usual pdf.

$$f_X(x | M) = \frac{dF_X(x | M)}{dx}$$

## Characteristic Function

- Definition

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{juX} f_X(x) dx$$

that is, the characteristic function of a random variable can be viewed as the Fourier transform of its probability density function.

- The pdf is then the inverse Fourier transform of the characteristic function

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-juX} \Phi_X(u) du$$

$$\frac{d}{du} \Phi(u) = \int_{-\infty}^{\infty} \left( \frac{d}{du} e^{juX} \right) f_X(x) dx = \int_{-\infty}^{\infty} jx e^{juX} f_X(x) dx$$

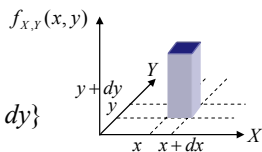
$$\left. \frac{d}{du} \Phi(u) \right|_{u=0} = \int_{-\infty}^{\infty} jx f_X(x) dx = j\bar{X}$$

$$\left. \frac{d^n}{du^n} \Phi(u) \right|_{u=0} = j^n E[X^n] = j^n \overline{X^n}$$

## Joint Probability Density Functions

Definition  $f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$

$$f_{X,Y}(x,y) dx dy = \Pr\{x < X \leq x + dx, y < Y \leq y + dy\}$$



- $f_{X,Y}(x,y) \geq 0, \quad -\infty < x < \infty, -\infty < y < \infty$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$
- $F_{X,Y}(x,y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(u,v) dv du$
- $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx \quad \rightarrow$  marginals
- $\Pr\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{X,Y}(x,y) dy dx$

## Expectation of Functions of Two R.V.s

- Similar definition as in single random variable case

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$$

- When  $g(X,Y) = X^n Y^k$

$E[X^n Y^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y) dx dy$  is called the **joint moment of X and Y**. When  $n=k=1$ , it is called correlation.  $n+k$  is the order of the moment.

$$\begin{aligned} E[(X - \bar{X})(Y - \bar{Y})] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{X})(y - \bar{Y}) f_{X,Y}(x,y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy - \bar{X}y - x\bar{Y} + \bar{X}\bar{Y}) f_{X,Y}(x,y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y) dx dy - \bar{X}\bar{Y} = \overline{XY} - \bar{X}\bar{Y} \end{aligned}$$

## Probability Distributions of Functions of Several Random Variables

- The probability space is defined on a hyperspace that contains the random variables.
- Function  $Y = g(X_1, X_2, \dots, X_N)$

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) = \Pr[g(X_1, X_2, \dots, X_N) \leq y] \\ &= \Pr\{\{\xi : g(x_1 = X_1(\xi), x_2 = X_2(\xi), \dots, x_N = X_N(\xi)) \leq y\}\} \end{aligned}$$

$$F_Y(y) = \Pr\{g(X_1, X_2, \dots, X_N) \leq y\} =$$

$$\int_{\{g(x_1, x_2, \dots, x_N) \leq y\}} f_{X_1, X_2, \dots, X_N}(x_1, x_2, \dots, x_N) dx_1 dx_2 \dots dx_N$$

$\implies$  Direct integration

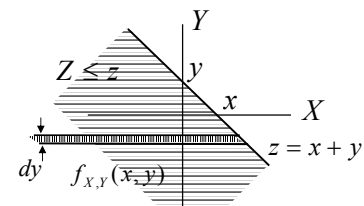
- Another method is through transformation of variables.

## Sum of Two Random Variables

$$Z = X + Y$$

At any  $y$ , the small stripe has a probability mass of

$$dy \int_{-\infty}^{z-y} f_{X,Y}(x,y) dx \longrightarrow dy f_{X,Y}(\bar{x}, y)$$



Thus, the shaded area which is  $\Pr\{Z \leq z\}$  can be obtained by

$$\Pr\{Z \leq z\} = F_Z(z) = \int_{-\infty}^z \int_{-\infty}^{z-y} f_{X,Y}(x,y) dx dy$$

If  $X$  and  $Y$  are independent,  $F_Z(z) = \int_{-\infty}^z f_Y(y) \int_{-\infty}^{z-y} f_X(x) dx dy$

And,  $f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y) dy$

Use Leibniz's rule

The pdf of the sum of two statistically independent random variables is the convolution of their individual pdf's.

## Probability Density Functions of Two R.V.s

$$Z = X / Y$$

Given  $z$ , the function  $x=yz$  is a line going through the origin.

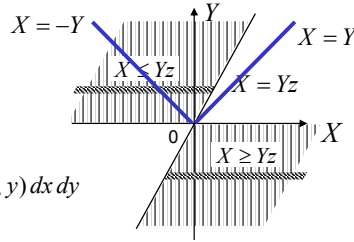
$$F_Z(z) = \Pr\{Z \leq z\} = \Pr\left\{\frac{X}{Y} \leq z\right\} = \iint_{\text{shaded area}} f_{X,Y}(x,y) dx dy$$

because if  $y > 0$ , then  $x \leq yz$ ; if  $y < 0$ , then  $x \geq yz$

$$F_Z(z) = \int_0^\infty \int_{-\infty}^{yz} f_{X,Y}(x,y) dx dy + \int_{-\infty}^0 \int_{yz}^\infty f_{X,Y}(x,y) dx dy$$

$$f_Z(z) = \int_0^\infty y f_{X,Y}(yz,y) dy - \int_{-\infty}^0 y f_{X,Y}(yz,y) dy = \int_{-\infty}^\infty |y| f_{X,Y}(yz,y) dy$$

If  $Y = g(X)$ , the integral for  $F_Z(z)$  becomes a line integral. A special case is when  $Y = |X|$ .



## Transformation of Multiple Random Variables

$Z = \varphi_1(X, Y)$  and  $W = \varphi_2(X, Y)$  are two functions of r.v.  $X$  and  $Y$ .

Both  $\varphi_1$  and  $\varphi_2$  are continuous functions with corresponding inverse functions  $X = \psi_1(Z, W)$  and  $Y = \psi_2(Z, W)$ , respectively.

Since all the events that map to  $\{x_1 < X(\xi) \leq x_2, y_1 < Y(\xi) \leq y_2\}$  would also map to  $\{z_1 < Z(\xi) \leq z_2, w_1 < W(\xi) \leq w_2\}$ , we have

$$\Pr\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = \Pr\{z_1 < Z \leq z_2, w_1 < W \leq w_2\}.$$

$$\text{or } \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{X,Y}(x,y) dy dx = \int_{z_1}^{z_2} \int_{w_1}^{w_2} f_{Z,W}(z,w) dw dz$$

$$\text{But } \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{X,Y}(x,y) dy dx = \int_{z_1}^{z_2} \int_{w_1}^{w_2} f_{X,Y}(\psi_1(z,w), \psi_2(z,w)) |J| dw dz$$

by way of change of variables with

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{\partial \psi_1}{\partial z} & \frac{\partial \psi_1}{\partial w} \\ \frac{\partial \psi_2}{\partial z} & \frac{\partial \psi_2}{\partial w} \end{vmatrix}$$

$J$  is the Jacobian that relates the incremental area  $dz dw$  to  $dx dy$ .

## Transformation of Multiple R.V. (cont'd)

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{X,Y}(x,y) dy dx = \int_{z_1}^{z_2} \int_{w_1}^{w_2} f_{X,Y}(\psi_1(z,w), \psi_2(z,w)) |J| dw dz$$

$$= \int_{z_1}^{z_2} \int_{w_1}^{w_2} f_{Z,W}(z,w) dw dz$$

$$\text{Therefore, } f_{Z,W}(z,w) = |J| f_{X,Y}[\psi_1(z,w), \psi_2(z,w)]$$

Example:  $Z = XY, W = X \Rightarrow X = W, Y = Z/W$

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ w^{-1} & \frac{z}{w^2} \end{vmatrix} = -w^{-1} \quad \text{Thus, } f_{Z,W}(z,w) = \frac{1}{|w|} f_{X,Y}\left(w, \frac{z}{w}\right)$$

$$\text{The marginals, } f_Z(z) = \int_{-\infty}^\infty \frac{1}{|w|} f_{X,Y}\left(w, \frac{z}{w}\right) dw$$

$$f_W(w) = \int_{-\infty}^\infty \frac{1}{|w|} f_{X,Y}\left(w, \frac{z}{w}\right) dz$$

## Characteristic Function of Sum of R.V.s

- The characteristic function of a random variable is the Fourier transform of its pdf.
- Very useful in deriving the moments of the r.v.

$$\Phi_X(u) = \int_{-\infty}^\infty f_X(x) e^{ju^x} dx \quad \text{and} \quad f_X(x) = (2\pi)^{-1} \int_{-\infty}^\infty \Phi_X(u) e^{-ju^x} du$$

$$\Phi_Y(u) = \int_{-\infty}^\infty f_Y(y) e^{ju^y} dy \quad \text{and} \quad f_Y(y) = (2\pi)^{-1} \int_{-\infty}^\infty \Phi_Y(u) e^{-ju^y} du$$

- Now,  $Z = X + Y$  where  $X$  and  $Y$  are independent,

The pdf of the sum of two statistically independent random variables is the convolution of their individual pdf's.

$$\Phi_Z(u) = \Phi_X(u) \Phi_Y(u)$$

$$f_Z(z) = (2\pi)^{-1} \int_{-\infty}^\infty \Phi_Z(u) e^{-ju^z} du = (2\pi)^{-1} \int_{-\infty}^\infty \Phi_X(u) \Phi_Y(u) e^{-ju^z} du$$

## Characteristic Function of A Gaussian R.V.

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{juX} f_X(x) dx$$

$$\Phi_X(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(jux) \exp\left[-\frac{(x-\bar{X})^2}{2\sigma^2}\right] dx$$

The exponential term:

$$juX - \frac{(x-\bar{X})^2}{2\sigma^2} = \frac{juX 2\sigma^2 - (x-\bar{X})^2}{2\sigma^2} = -\frac{[x - (\bar{X} + ju\sigma^2)]^2}{2\sigma^2} + ju\bar{X} - \frac{u^2\sigma^2}{2}$$

$$\begin{aligned} \Phi_X(u) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{[x - (\bar{X} + ju\sigma^2)]^2}{2\sigma^2}\right] + ju\bar{X} - \frac{u^2\sigma^2}{2} dx \\ &= \left\{ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{[x - (\bar{X} + ju\sigma^2)]^2}{2\sigma^2}\right] dx \right\} \exp\left(ju\bar{X} - \frac{u^2\sigma^2}{2}\right) \end{aligned}$$

$$\text{because } \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{[x - (\bar{X} + ju\sigma^2)]^2}{2\sigma^2}\right] dx = 1$$

Use Cauchy theorem for contour integral – net result is that it is just the same as integrating along the real axis.

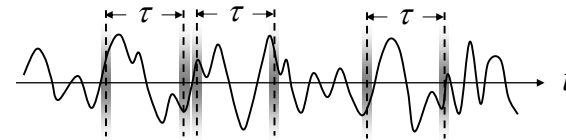
## Wide Sense Stationary Processes

$$R_X(t_1, t_2) = R_X(t_1 + T, t_2 + T) = E[X(t_1 + T)X(t_2 + T)]$$

$$\text{Set } T = -t_1 \Rightarrow R_X(t_1, t_2) = R_X(t_1 - T, t_2 - T) = R_X(0, t_2 - t_1)$$

For a wide sense stationary process, the autocorrelation function does not depend on the absolute time origin. The first argument 0 is thus arbitrary and the autocorrelation is a function of only the time difference,  $t_2 - t_1 = \tau$ .

$$R_X(\tau) = R_X(0, t_2 - t_1) = E[X(t_1)X(t_1 + \tau)] = E[X(t)X(t + \tau)]$$



The darkness represents the height of  $f_X(x)$

If we look at any two time instances of a wide sense stationary process, their correlation is only a function of their time difference, no matter where they are. In subsequent discussions, wide sense stationarity is always assumed.

## Properties of Autocorrelation Functions (I)

- $R_X(0) = \overline{X^2} \geq 0$
- Symmetry:  $R_X(\tau) = R_X(-\tau)$ 

$$R_X(\tau) = E[X(t)X(t + \tau)] = E[X(t - \tau)X(t)] = R_X(-\tau)$$
- $|R_X(\tau)| \leq R_X(0)$ 

$$E[(X_1 \pm X_2)^2] = E[X_1^2 \pm 2X_1X_2 + X_2^2] \geq 0$$

$$E[X_1^2 + X_2^2] = 2R_X(0) \geq |E[2X_1X_2]| = 2|R_X(\tau)|$$
- If  $X(t)$  has a constant component, say,  $X(t) = A + V(t)$ , where  $V(t)$  has zero mean, the autocorrelation function has a constant component.

$$\begin{aligned} E\{[A + V(t)][A + V(t + \tau)]\} &= E\{A^2 + AV(t) + AV(t + \tau) + V(t)V(t + \tau)\} \\ &= A^2 + AE[V(t)] + AE[V(t + \tau)] + E[V(t)V(t + \tau)] = A^2 + R_V(\tau) \end{aligned}$$

## Properties of Autocorrelation Functions (II)

- If  $X(t)$  has a periodic component, then the autocorrelation function has a periodic component.

$$X(t) = A \cos(\omega t + \Theta)$$

where  $A$  and  $\omega$  are constant and  $\Theta$  a r.v. uniformly distributed over  $(0, 2\pi)$ ; i.e.  $f_\Theta(\theta) = (2\pi)^{-1}$ ,  $0 \leq \theta < 2\pi$ ;  $= 0$ , elsewhere.

$$\begin{aligned} R_X(\tau) &= E[A \cos(\omega t + \Theta) A \cos(\omega t + \omega \tau + \Theta)] \\ &= E\left[\frac{A^2}{2} \cos(2\omega t + \omega \tau + 2\Theta) + \frac{A^2}{2} \cos(\omega \tau)\right] \\ &= \frac{A^2}{2} \cos(\omega \tau) + \frac{A^2}{2} \int_0^{2\pi} \frac{1}{2\pi} \cos(2\omega t + \omega \tau + 2\theta) d\theta = \frac{A^2}{2} \cos(\omega \tau) \end{aligned}$$

- If  $X(t)$  is ergodic and zero-mean, and has no periodic components,

$$\lim_{|\tau| \rightarrow \infty} R_X(\tau) = 0$$

That is, time samples far apart tend to behave statistically independently.

## Properties of Autocorrelation Functions (III)

- Autocorrelation functions cannot have arbitrary shape – they must correspond to some power spectrum which must be non-negative over the entire frequency range. More discussions later.

$$F [R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau = \text{power spectrum of } X(t)$$

$$F [R_X(\tau)] = S_X(\omega) \geq 0 \quad \text{for all } \omega$$

Example: An ergodic random process has an autocorrelation function of the form

$$R_X(\tau) = \frac{4\tau^2 + 6}{\tau^2 + 1}$$

Find the mean-square value, mean value, and variance of the process.

$$R_X(\tau) = \frac{4\tau^2 + 6}{\tau^2 + 1} = 4 + \frac{2}{\tau^2 + 1} \quad \overline{X^2} = R_X(0) = 6$$

$$\overline{X^2} = 4 \Rightarrow \overline{X} = \pm 2 \quad \sigma_X^2 = \overline{X^2} - (\overline{X})^2 = 6 - 4 = 2$$

## Sinusoid Plus Noise

Let  $X(t) = A \cos(\omega t + \Theta)$  where  $\Theta$  is a random variable uniformly distributed over  $(0, 2\pi)$ .

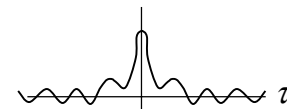
$$R_X(\tau) = \frac{1}{2} A^2 \cos \omega \tau$$

Let  $V(t)$  be a zero mean noise process, statistically independent of  $X(t)$ , the signal, with autocorrelation function:

$$R_V(\tau) = B^2 e^{-\alpha|\tau|}$$

The observed process is  $Z(t) = A \cos(\omega t + \Theta) + V(t)$  which has an autocorrelation function:

$$R_Z(\tau) = \frac{1}{2} A^2 \cos \omega \tau + B^2 e^{-\alpha|\tau|}$$



Note that  $R_V(\tau) = B^2 e^{-\alpha|\tau|} \rightarrow 0$ , as  $\tau \rightarrow \infty$ .

It is thus possible to recover a sinusoid from noise contamination as long as we measure the autocorrelation at sufficiently long time lags.

## Spectral Density

$$S_X(\omega) = \lim_{T \rightarrow \infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T}$$

$$\overline{X^2} = R_X(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

Since  $S_X(\omega)$  is an average over time, it is thus usually called a **power density spectrum**. When  $S_X(\omega)$  is integrated over the entire frequency range, we obtain the average power of the signal, which is equal to the mean-square value of the wide-sense stationary process.

Example:  $S_X(\omega) = \frac{2a}{\omega^2 + a^2} \quad S_X(0) = \frac{2a}{0^2 + a^2} = \frac{2}{a}$

$$\overline{X^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{\omega^2 + a^2} d\omega = \frac{1}{2\pi} \left[ \frac{2a}{a} \tan^{-1}\left(\frac{\omega}{a}\right) \right]_{-\infty}^{\infty} = \frac{2}{2\pi} \left[ \frac{\pi}{2} + \frac{\pi}{2} \right] = 1$$

Useful integral:

$$\frac{1}{2\pi} \int_p^q \frac{2a}{\omega^2 + a^2} d\omega = \frac{1}{\pi} \left[ \tan^{-1}\left(\frac{\omega}{a}\right) \right]_p^q$$

$\omega$  in radian/s

$$\int_p^q \frac{2a}{(2\pi f)^2 + a^2} df = \frac{1}{\pi} \left[ \tan^{-1}\left(\frac{2\pi f}{a}\right) \right]_p^q$$

$f$  in cycle/s or Hz

## Spectral Density of Constant or Periodic Signals

Consider a process  $X(t) = A + B \cos(2\pi f_0 t + \Theta)$

where  $A$ ,  $B$  and  $f_0$  are constant and  $\Theta$  is a random variable uniformly distributed over  $(0, 2\pi)$ .

Let  $X_T(t)$  be a truncated version of  $X(t)$  over  $(-T, T)$ .

$$F_{X_T}(f) = \int_{-T}^T [A + B \cos(2\pi f_0 t + \Theta)] e^{-j2\pi f t} dt$$

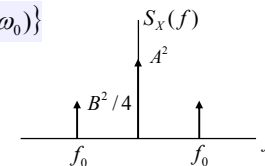
$$E[F_{X_T}(f)] = \int_{-\infty}^{\infty} [A + B \cos(2\pi f_0 t + \Theta)] e^{-j2\pi f t} dt$$

$$= A\delta(f) + (B/2) [\delta(f + f_0) e^{-j\Theta} + \delta(f - f_0) e^{j\Theta}]$$

$$S_X(f) = A^2 \delta(f) + (B^2/4) \{ \delta(f + f_0) + \delta(f - f_0) \}$$

$$S_X(\omega) = 2\pi A^2 \delta(\omega) + (\pi B^2/2) \{ \delta(\omega + \omega_0) + \delta(\omega - \omega_0) \}$$

The spectral density thus consists of three spikes (delta function) at DC (with height  $A^2$ ) and at  $\pm f_0$  (with height  $B^2/4$ ), respectively.



## Mean-Square Value and Total Power

$$X(t) = A + B \cos(\omega_0 t + \Theta) \quad \Theta \text{ uniformly distributed in } (0, 2\pi)$$

$$S_X(\omega) = 2\pi A^2 \delta(\omega) + (\pi B^2 / 2) \{ \delta(\omega + \omega_0) + \delta(\omega - \omega_0) \}$$

Total power:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ 2\pi A^2 \delta(\omega) + (\pi B^2 / 2) [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \} d\omega \\ &= A^2 + \frac{B^2}{4} + \frac{B^2}{4} = A^2 + \frac{B^2}{2} \end{aligned}$$

Mean-Square Value of the process

$$E_{\Theta} [X^2(t)] = E_{\Theta} \{ [A + B \cos(\omega_0 t + \Theta)]^2 \} = E_{\Theta} \{ A^2 + 2AB \cos(\omega_0 t + \Theta) + B^2 \cos^2(\omega_0 t + \Theta) \}$$

$$= A^2 + E_{\Theta} \{ 2AB \cos(\omega_0 t + \Theta) \} + B^2 E_{\Theta} \{ \cos^2(\omega_0 t + \Theta) \}$$

$$E_{\Theta} \{ 2AB \cos(\omega_0 t + \Theta) \} = 0 \quad \text{for } \Theta \sim \mathcal{U}(0, 2\pi)$$

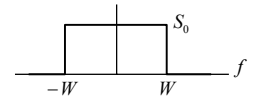
$$E_{\Theta} \{ \cos^2(\omega_0 t + \Theta) \} = E_{\Theta} \left\{ \frac{\cos(2\omega_0 t + 2\Theta) + 1}{2} \right\} = \frac{1}{2}$$

$$\text{Therefore, } E_{\Theta} [X^2(t)] = \overline{X^2} = A^2 + \frac{B^2}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

## Autocorrelation of Bandlimited White Noise

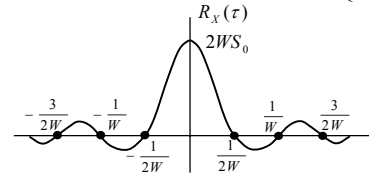
A more useful concept is the bandlimited white noise whose spectral density is a constant over a finite bandwidth and zero outside the frequency range. For example:

$$S_X(\omega) = \begin{cases} S_0, & |\omega| \leq 2\pi W \\ 0, & |\omega| > 2\pi W \end{cases}$$



$$R_X(\tau) = \mathbf{F}^{-1} \{ S_X(f) \} = \mathbf{F}^{-1} \left\{ S_0 \text{rect} \left( \frac{f}{2W} \right) \right\} = 2WS_0 \text{sinc}(2W\tau)$$

$$R_X(\tau) = 0 \quad \text{at } \tau = n/2W, \quad n = \pm 1, \pm 2, \dots$$



Random variables from a bandlimited white noise are uncorrelated if they are separated in time by any multiple of  $1/2W$  seconds.

Therefore, if a continuous time bandlimited white noise process is sampled at twice the maximum frequency limit ( $2W$ ), then the resultant samples of the discrete time sequence are uncorrelated.

## Spectral Density of Binary Processes

$$S_X(\omega) = \lim_{T \rightarrow \infty} \frac{E[|F_{X_T}(\omega)|^2]}{2T} = \frac{A^2 |F_p(\omega)|^2}{t_1}$$

For  $p(t) = p_r(t) = \text{rect}(t/t_1)$ ,

$$F_p(\omega) = t_1 \text{sinc}(t_1 \omega / 2\pi) \quad \text{and thus } |F_p(\omega)|^2 = t_1^2 \text{sinc}^2(t_1 \omega / 2\pi)$$

$$\text{Therefore, } S_X(\omega) = \frac{A^2 |F_p(\omega)|^2}{t_1} = A^2 t_1 \text{sinc}^2(t_1 \omega / 2\pi)$$

$$\text{For } p(t) = p_c(t) = \frac{1}{2} \left( 1 + \cos \frac{2\pi t}{t_1} \right), \quad |t| \leq \frac{t_1}{2}; = 0, \quad |t| > \frac{t_1}{2},$$

$$F_p(\omega) = \frac{1}{2} \int_{-t_1/2}^{t_1/2} \left( 1 + \cos \frac{2\pi t}{t_1} \right) e^{-j\omega t} dt = \frac{t_1}{2} \left[ \frac{\sin(\omega t_1 / 2)}{(\omega t_1 / 2)} \right] \left[ \frac{\pi^2}{\pi^2 - (\omega t_1 / 2)^2} \right]$$

$$S_X(\omega) = \frac{A^2 t_1}{4} \left[ \frac{\sin(\omega t_1 / 2)}{(\omega t_1 / 2)} \right]^2 \left[ \frac{\pi^2}{\pi^2 - (\omega t_1 / 2)^2} \right]^2$$

$$S_X(f) = \frac{A^2 t_1}{4} \text{sinc}^2(t_1 f) \left[ \frac{1}{1 - (t_1 f)^2} \right]^2$$

Note that in both cases,  $\max S_X(\omega)$  occurs at  $\omega = 0$

## Random Input to A System



$$x(t) \Rightarrow X(t) \quad y(t) \Rightarrow Y(t)$$

The input is no longer a fixed function.

The same linear system concept applies.

$$\text{Example: } h(t) = \begin{cases} 5e^{-3t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$X(t) = M + 4 \cos(2t + \Theta)$  where  $M$  is a random variable and  $\Theta$  is an independent random variable, uniformly distributed in  $(0, 2\pi)$ .

$$\begin{aligned} Y(t) &= \int_{-\infty}^t [M + 4 \cos(2\lambda + \Theta)] 5e^{-3(t-\lambda)} d\lambda \\ &= \frac{5}{3} M + \frac{20}{13} [3 \cos(2t + \Theta) + 2 \sin(2t + \Theta)] \end{aligned}$$

$Y(t)$  is also a random process whose statistical properties can be derived from the distributions of the random variables,  $M$  and  $\Theta$ .



## Example

A linear system has an impulse response of the form

$$h(t) = \begin{cases} 5\delta(t) + 3, & 0 \leq t < 1 \\ 0, & \text{elsewhere} \end{cases}$$

The input process is of the form  $X(t) = 2 \cos(2\pi t + \Theta)$ ,  $-\infty < t < \infty$  where  $\Theta$  is a random variable uniformly distributed in  $(0, 2\pi)$ .

1. Write an expression for the output process;
2. Find the mean value of the output;
3. Find the variance of the output.

$$Y(t) = \int_0^1 2 \cos[2\pi(t - \lambda) + \Theta][5\delta(\lambda) + 3]d\lambda$$

$$= 10 \cos(2\pi t + \Theta) - \frac{6}{2\pi} \sin[2\pi(t - \lambda) + \Theta] \Big|_{\lambda=0}^1 = 10 \cos(2\pi t + \Theta)$$

$$E[Y(t)] = E_{\Theta}[10 \cos(2\pi t + \Theta)] = 0$$

$$E[Y^2(t)] = E_{\Theta}[100 \cos^2(2\pi t + \Theta)] = E_{\Theta}[50 + 50 \cos(4\pi t + 2\Theta)] = 50$$

$$\sigma_Y^2 = E[Y^2(t)] - \{E[Y(t)]\}^2 = 50 - 0 = 50$$

## Mean of System Output

As demonstrated in the previous examples,

$$\bar{Y} = E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(t - \lambda)h(\lambda)d\lambda\right]$$

In general, for  $E\left[\int_{t_1}^{t_2} Z(t)f(t)dt\right] = \int_{t_1}^{t_2} E[Z(t)]f(t)dt$ , it requires that

1.  $\int_{t_1}^{t_2} E[|Z(t)|] |f(t)| dt$
2.  $Z(t)$  is bounded on the interval  $(t_1, t_2)$ .

In most cases, we assume these conditions are satisfied.

And if  $X(t)$  is wide sense stationary with  $E[X(t)] = \bar{X}$ , then

$$\bar{Y} = \int_{-\infty}^{\infty} E[X(t - \lambda)]h(\lambda)d\lambda = \bar{X} \int_{-\infty}^{\infty} h(\lambda)d\lambda$$

Note that  $\int_{-\infty}^{\infty} h(\lambda)d\lambda$  is the dc gain of the system; the dc component of the output is thus equal to the dc component of the input times the dc gain of the system.

## Mean-Square Value of System Output

$$\bar{Y}^2 = E[Y^2(t)] = E\left[\int_0^{\infty} X(t - \lambda_1)h(\lambda_1)d\lambda_1 \cdot \int_0^{\infty} X(t - \lambda_2)h(\lambda_2)d\lambda_2\right]$$

$$= E\left[\int_0^{\infty} d\lambda_1 \int_0^{\infty} X(t - \lambda_1)X(t - \lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_2\right]$$

$$= \int_0^{\infty} d\lambda_1 \int_0^{\infty} E[X(t - \lambda_1)X(t - \lambda_2)]h(\lambda_1)h(\lambda_2)d\lambda_2$$

But,  $E[X(t - \lambda_1)X(t - \lambda_2)] = R_X(t - \lambda_1 - t + \lambda_2) = R_X(\lambda_2 - \lambda_1)$

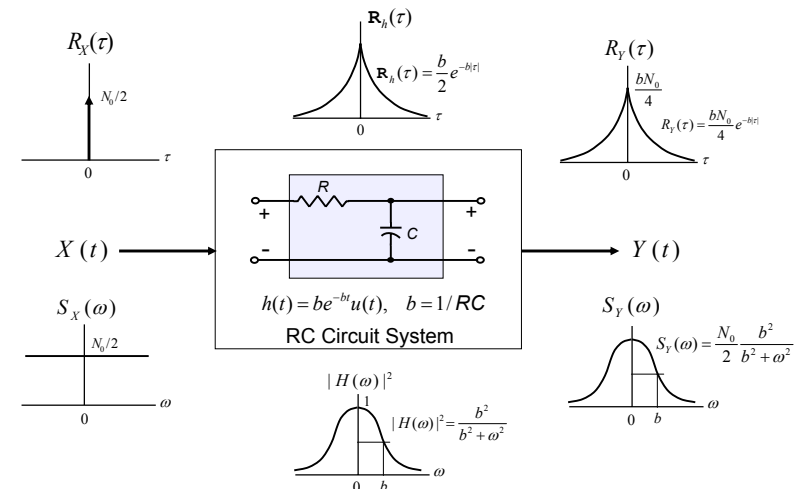
Therefore,  $\bar{Y}^2 = \int_0^{\infty} d\lambda_1 \int_0^{\infty} R_X(\lambda_2 - \lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2$

Example: For white noise input with  $R_X(\tau) = \frac{N_0}{2} \delta(\tau)$ , the output "noise" power is

$$\bar{Y}^2 = \int_0^{\infty} d\lambda_1 \int_0^{\infty} R_X(\lambda_2 - \lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2$$

$$= \int_0^{\infty} d\lambda_1 \int_0^{\infty} \frac{N_0}{2} \delta(\lambda_2 - \lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 = \frac{N_0}{2} \int_0^{\infty} h^2(\lambda)d\lambda$$

## RC Circuit – Input Output Relationship



# Finite-Time Integrator – Input Output Relationship

