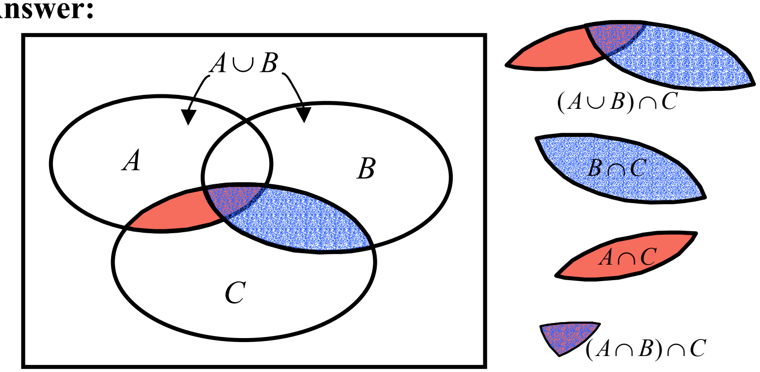
Problem #1 (10 pts.)

Use Venn diagram to show that $\Pr(C \cap (A \cup B)) \le \Pr(C \cap A) + \Pr(C \cap B)$

Answer:



$C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$

$\Pr(C \cap (A \cup B)) = \Pr(C \cap A) + \Pr(C \cap B) - \Pr(C \cap A \cap B)$

Since $\Pr(C \cap A \cap B) \ge 0$,

$\Pr(C \cap (A \cup B)) \le \Pr(C \cap A) + \Pr(C \cap B)$

The equality holds iff $C \cap A \cap B = \phi$.

Problem #2 (30 pts, 15 pts each)

Two solid-state diodes are to be connected into a new circuit. Each diode has a probability of 0.02 that it will fail as a short circuit and a probability of 0.01 as an open circuit. We assume that these two diodes are independently obtained.

1. If the two diodes are connected in series in the same direction, find the probability that the new circuit will function as a diode;
2. If the two diodes are connected in parallel in the same direction, find the probability that the new circuit will NOT function as a diode.

Answer:

1. Pr{2 diode in series function as a diode} = Pr{2 good diode}+Pr{1 good diode, 1 short-circuit diode}
   = 0.97*0.97+2*0.97*0.02 = 0.9409+0.0388 = 0.9797

2. Pr{2 diode in parallel function as a diode} = Pr{2 good diode}+Pr{1 good diode, 1 open-circuit diode}
   = 0.97*0.97+2*0.97*0.01 = 0.9409+0.0194 = 0.9603

Pr{2 diode in parallel NOT function as a diode} = 1 - Pr{2 diode in parallel function as a diode} = 1 - 0.9603 = 0.0397

Problem #3 (20 pts)

Random variable X has probability density function of the form: $f_X(x) = 0.2[u(x+1) - u(x-4)]$.
Random variable Y is a function of X, $Y = 4(X+1)^2$. Find $f_Y(y)$.

Answer:

Let $Z = X + 1$. Since it is a monotonic function and $\frac{dx}{dz} = 1$, $f_Z(z) = f_X(x)$ within the corresponding range of x.

That is, $-1 \le x \le 4 \Leftrightarrow 0 \le z \le 5$. Now, $y = 4z^2 \Rightarrow z = \frac{\sqrt{y}}{2}$, for $0 \le y \le 100$, because $0 \le z \le 5$.

With $\frac{dz}{dy} = \frac{1}{4\sqrt{y}}$ (except at $y = 0$), therefore, $f_Y(y) = 0.2 \bullet \frac{1}{4\sqrt{y}} = \frac{1}{20\sqrt{y}}$, for $0 < y \le 100$ and $= 0$, elsewhere.

Note, $\int_0^{100} \frac{1}{20\sqrt{y}}\, dy = 1$

Problem #4 (20 pts, 10 pts each)

Consider an experiment in which a single card is drawn from a deck of cards.

1. If a usual 52-card deck is used, are the event of drawing a spade and that of drawing a ten independent? Provide proof to justify your answer.
2. If instead, a 54-card deck with jokers is used, are the same two events independent? Provide proof to justify your answer.

Answer:

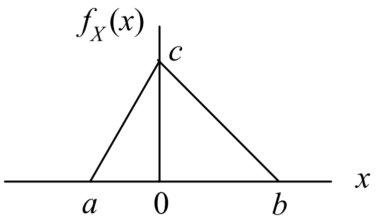
1. When 52 cards are used, Pr{spade} = 13/52, Pr{ten} = 4/52, and Pr{ten-spade} = 1/52.
   Since Pr{spade} * Pr{ten} = (13/52)*(4/52) = 1/52 = Pr{ten-spade}, the two events {spade} and {ten} are independent.

2. When 54 cards are used, Pr{spade} = 13/54, Pr{ten} = 4/54, and Pr{ten-spade} = 1/54.
   Since Pr{spade} * Pr{ten} = (13/54)*(4/54) $\neq$ 1/54 = Pr{ten-spade}, the two events {spade} and {ten} are NOT independent.

Problem #5 (20 pts, 10 pts each; 3 is extra)

The probability density function of r.v. X is as plotted.

1. Find c in terms of a and b such that it is a valid pdf.
2. Find the mean and variance of X;
3. Find the mean and variance of Y, $Y = |X|$



Answer:

1. $$\int_a^b f_X(x)\,dx = \frac{1}{2}(-ac+bc) = 1 \qquad \therefore\ c = \frac{2}{b-a}$$

2. $$\overline{X} = \int_a^b x f_X(x)\,dx = \int_a^0 x\frac{c}{a}(-x+a)\,dx + \int_0^b x\frac{c}{b}(-x+b)\,dx = -\frac{ca^2}{6} + \frac{cb^2}{6} = \frac{2}{b-a}\,\frac{b^2-a^2}{6} = \frac{b+a}{3}$$

$$\overline{X^2} = \int_a^b x^2 f_X(x)\,dx = \int_a^0 x^2\frac{c}{a}(-x+a)\,dx + \int_0^b x^2\frac{c}{b}(-x+b)\,dx = -\frac{ca^3}{12} + \frac{cb^3}{12} = \frac{1}{6}(b^2+ab+a^2)$$

$$\sigma^2 = \overline{X^2} - \overline{X}^2 = \frac{1}{6}(b^2+ab+a^2) - \left(\frac{b+a}{3}\right)^2 = \frac{b^2+ab+a^2}{6} - \frac{b^2+2ab+a^2}{9} = \frac{b^2-ab+a^2}{18}$$

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