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ECE 3075
HW #4
Solution

Problem 4.1 (2-3.3)

(a) $f_X(x) = \exp(-2|x|)$, $-\infty < x < \infty$

$Y = X^2 \Rightarrow X_{1,2} = \pm\sqrt{Y} \Rightarrow$ 2 roots for $Y > 0$

no solution for $Y > 0$

$$f_Y(y) = f_X(+\sqrt{y})\frac{1}{\left|\frac{dY}{dx}\right|_{+\sqrt{y}}} + f_X(-\sqrt{y})\frac{1}{\left|\frac{dY}{dx}\right|_{-\sqrt{y}}} = \exp(-2\sqrt{y})\frac{1}{|+2\sqrt{y}|} + \exp(-2\sqrt{y})\frac{1}{|-2\sqrt{y}|}$$

$$= \boxed{\exp(-2\sqrt{y})\cdot\frac{1}{\sqrt{y}} \quad , y > 0}$$

we have $\boxed{f_Y(y) = 0 \text{ for } y < 0}$, since Y can never be negative.

(b) we can solve the problem in two ways:

(i) (Easier) $\Pr\{Y > 2\} = \Pr\{|X| > \sqrt{2}\} = 1 - \int_{-\sqrt{2}}^{\sqrt{2}} \exp(-2|x|)dx = 1 - 2\int_0^{\sqrt{2}} \exp(-2x)dx$

$$= 1 - 2\left(\tfrac{-1}{2}\right)\left(\exp(-2\sqrt{2}) - 1\right)$$

$$= \boxed{\exp(-2\sqrt{2})}$$

(ii) (Equivalent, but trickier) $\Pr\{Y > 2\} = \int_2^\infty f_Y(y)dy = \int_2^\infty \exp(-2\sqrt{y})\frac{1}{\sqrt{y}}dy$

let $z = 2\sqrt{y} \Rightarrow dz = \frac{1}{\sqrt{y}}dy$

$$\Rightarrow \Pr\{Y > 2\} = \int_{2\sqrt{2}}^\infty \exp(-z)dz = \boxed{\exp(-2\sqrt{2})}$$

Problem 4.2 (2-4.3)

(a) $\int_0^6 f(y)dy = \int_0^6 Ky\,dy = \frac{K}{2}\left[y^2\right]_0^6 = \frac{K}{2}\cdot 36 = 1 \Rightarrow \boxed{K = \frac{1}{18}}$

(b) $E[Y] = \bar{Y} = \int_0^6 y f_Y(y)dy = \int_0^6 y\left(\frac{1}{18}y\right)dy = \frac{1}{18}\frac{y^3}{3}\Big|_0^6 = \boxed{4}$

(c) $E[Y^2] = \int_0^6 y^2 f_Y(y)\,dy = \int_0^6 y^2\left(\frac{1}{18}y\right)dy = \frac{1}{18}\frac{y^4}{4} = \boxed{18}$

(d) $var[Y] = E[(Y-\bar{Y})^2] = E[Y^2] - \bar{Y}^2 = 18 - 16 = \boxed{2}$

(e) $E[(Y-\bar{Y})^3] = \int_0^6 (y-4)^3 f_Y(y)\,dy = \int_0^6 (y-4)^3 (\frac{1}{18} y)\,dy = \frac{1}{18} \int_0^6 (y^4 - 12y^3 + 48y^2 - 64y)\,dy$

$= \boxed{-1{,}6}$

Problem 4.3 (2-5.1) $f_V(v) = \frac{1}{\sqrt{2\pi}.4} e^{-\frac{(v-5)^2}{2.16}}$

(a) $Pr\{V > 0\} = Q\left(\frac{0-5}{4}\right) = Q(-1{,}25) = \boxed{0{,}8944}$

(b) $Pr\{0 < V < 5\} = Q\left(\frac{0-5}{4}\right) - Q\left(\frac{5-5}{4}\right) = 0{,}8944 - \frac{1}{2} = \boxed{0{,}3944}$

(c) $Pr\{V > 10\} = Q\left(\frac{10-5}{4}\right) = \boxed{0{,}1057}$

Problem 4.4 (2-6.1)

(a) Instantaneous power: $P = I^2 R$ (random variable because I is an r.v.)

Mean power: $\bar{P} = E[I^2 R] = R.E[I^2] = R(\sigma_I^2 + \mu_I^2)$

$= R(4+0) = 3.4$

$= \boxed{12}$ (W)

(b) $var[P] = E[P^2] - \bar{P}^2 = E[R^2 I^4] - \bar{P}^2 = R^2 E[I^4] - \bar{P}^2$

From (2-27) in textbook,

$E[I^4] = 3\sigma_I^4 + 6\sigma_I^2(\mu_I)^2 + (\mu_I)^4 = 3\sigma_I^4 = 3.4^2 = \underline{\underline{48}}$

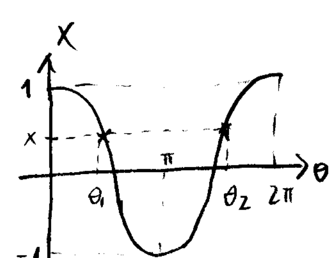
$\Rightarrow var[P] = 3^2.48 - 12^2 = \boxed{288}$

(c) $Pr\{P > 36\} = Pr\{I^2 > 12\} = Pr\{|I| > 2\sqrt{3}\} = 2Q\left(\frac{2\sqrt{3}-0}{2}\right) = 2Q(\sqrt{3}) = \boxed{0{,}0832}$

Problem 4.5 (2-7.1) $\frac{1}{\left|\frac{dx}{d\theta}\right|} = \frac{1}{|-\sin\theta|} = \frac{1}{|\sin\theta|} = \frac{1}{\sqrt{1-\cos^2\theta}} = \frac{1}{\sqrt{1-x^2}}$

(a) Inverting $X = \cos\theta$, we find that there are two θ values corresponding to each $X \in [-1,1]$:

$\theta_{1,2} = \cos^{-1} x,\ 2\pi - \cos^{-1} x$



(a) (Cont'd)

$$\Rightarrow f_X(x) = \left(f_\Theta(\cos^{-1}x) + f_\Theta(2\pi - \cos^{-1}x)\right)\frac{1}{\sqrt{1-x^2}} = \left(\frac{1}{2\pi} + \frac{1}{2\pi}\right)\frac{1}{\sqrt{1-x^2}} = \boxed{\frac{1}{\pi\sqrt{1-x^2}}, \quad -1 \le x \le 1}$$

$\boxed{f_X(x) = 0 \text{ for } |x| > 1}$ because $-1 \le \cos\theta = x \le 1$.

(b) $$E[X] = E[\cos\theta] = \int_0^{2\pi} \cos\theta \, f_\theta(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, d\theta \boxed{= 0}$$

(c) $$\text{var}[X] = E[X^2] - E^2[X] = \int_0^{2\pi} \cos^2\theta \, f_\theta(\theta)\, d\theta - 0^2 = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{1}{2\pi}\left(\frac{\theta}{2} + \frac{1}{2}\sin 2\theta\right)_0^{2\pi}$$

$$= \boxed{\frac{1}{2}}$$

(d) $$\Pr[X > 0.5] = \int_{0.5}^{1} f_X(x)\, dx = \int_{0.5}^{1} \frac{1}{\pi\sqrt{1-x^2}}\, dx$$

let $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta \Rightarrow \int \frac{1}{\pi\sqrt{1-x^2}}\, dx = \frac{-1}{\pi}\int \frac{\cancel{\sin\theta}}{\cancel{\sin\theta}}\, d\theta = \frac{-\theta}{\pi} = \frac{-\cos^{-1}x}{\pi}$

$$\Rightarrow \int_{0.5}^{1} \frac{1}{\pi\sqrt{1-x^2}}\, dx = \frac{-1}{\pi}\left[\cos^{-1}x\right]_{0.5}^{1} = \frac{-1}{\pi}\left[0 - \frac{\pi}{3}\right] = \boxed{\frac{1}{3}}$$