ECE 3075A
Random Signals

Lecture 11
Monte Carlo Simulation, Characteristics Functions

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Conditional Probability Distribution

We define the conditional probability the same as before.

$$F_X(x\,|\,M) = \Pr(X \le x\,|\,M) = \frac{\Pr(X \le x, M)}{\Pr(M)}, \quad \Pr(M) > 0$$

If we use the event mapping concept, $\Pr(X \le x, M)$ is the probability of all the outcomes which realize both events $X(\xi) \le x$ and $\xi \in M$.

If $M = \{X \le m\}$, $F_X(x\,|\,M) = \Pr(X \le x\,|\,X \le m) = \dfrac{\Pr(X \le x, X \le m)}{\Pr(X \le m)}$

If $x \le m$, $F_X(x\,|\,M) = \dfrac{\Pr(X \le x)}{\Pr(X \le m)} = \dfrac{F_X(x)}{F_X(m)}$ If $x \ge m$, $F_X(x\,|\,M) = \dfrac{\Pr(X \le m)}{\Pr(X \le m)} = 1$

Conditional probability density function has all the properties of a usual pdf. $f_X(x\,|\,M) = \dfrac{dF_X(x\,|\,M)}{dx}$

Conditional Expected Value

- The expected value of X, given event M, is

$$E[X\,|\,M] = \int_{-\infty}^{\infty} x f_X(x\,|\,M)\,dx$$

if $M = \{X \le m\}$, $f_X(x\,|\,X \le m) = \begin{cases} \dfrac{f_X(x)}{F_X(m)}, & x < m \\ 0, & x \ge m \end{cases}.$

Thus, $E[X\,|\,X \le m] = \dfrac{\int_{-\infty}^{m} x f_X(x)\,dx}{\int_{-\infty}^{m} f_X(x)\,dx}$

which is the expected value of X when X is constrained to the set/event $\{X \le m\}$.

Useful & Intriguing Inequalities

- Chebychev Inequality

$$\Pr\{|X - \bar{X}| \ge \varepsilon\} \le \sigma_X^2 / \varepsilon^2$$

$$\Pr\{|X - \bar{X}| \ge \varepsilon\} = \int_{-\infty}^{\bar{X}-\varepsilon} f_X(x)\,dx + \int_{\bar{X}-\varepsilon}^{\infty} f_X(x)\,dx = \int_{x \in B} f_X(x)dx$$

where $B = \left\{R - \{|x - \bar{X}| < \varepsilon\}\right\}$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x)\,dx \ge \int_{x \in B} (x - \bar{X})^2 f_X(x)\,dx \ge \varepsilon^2 \int_{x \in B} f_X(x)\,dx$$
$$= \varepsilon^2 \Pr\{|X - \bar{X}| \ge \varepsilon\}$$

- Markov Inequality: for non-negative r.v. X,

$$\Pr\{X \ge a\} \le E[X]/a, \quad a > 0$$

Monte Carlo Simulation

- To generate a random variable with a prescribed probability distribution function, often needed in computer simulation of a certain physical phenomenon such as noise in a measurement;
- Construction of such a simulation begins with a random variable that has uniform distribution on [0,1];
- Find a monotonically non-decreasing transformation $Y=T(X)$ such that Y has the prescribed distribution $F_Y(y)$.

$$\Pr\{Y \le y\} = F_Y\left[y = T(x)\right] = F_X(x) = \Pr\{X \le x\}$$

Since X is uniformly distributed, $F_X(x) = x$ for $0 \le x \le 1$.

Transformation $y = T(x) = F_Y^{-1}(x)$, for $0 \le x \le 1$ will produce the desired simulation of r.v. Y.

Uniform Random Number Generator

- Algorithmic generation of *random* numbers that display uniform distribution between 0 and 1;
- No algorithm can produce true random numbers; they are called pseudo-random numbers instead;
- Integer linear congruence method is often used:

$$x_{n+1} = ((ax_n + c))_m = (ax_n + c)_{\text{mod } m}$$

where a and c are certain fixed values and x_0 is an initial number known as the seed. Normalizing the result by m will produce a pseudo-random number sequence with value in [0,1). Such a sequence is periodic with a "cycle" less than or equal to m.

Example of Monte Carlo Simulation

Let $a = 9, c = 5, m = 16,$ and $x_0 = 4.$

$$x_{n+1} = (9x_n + 5)_{\text{mod } 16}$$

Try $a = 9, c = 3, m = 16,$ and $x_0 = 4,$ and see what happens.

Integer sequence: 4, 9, 6, 11, 8, 13, 10, 15, 12, 1, 14, 3, 0, 5, 2, 7, 4, ….

Normalized plus an epsilon (0.0001):
0.2501, 0.5626, 0.3751, 0.6876, 0.5001, 0.8126, 0.6251, 0.9376, 0.7501, 0.0626, 0.8751, 0.1876, 0, 0.3126, 0.1251, 0.4376, 0.2501, …

E.g., Desired Probability Distribution Function: $F_Y(y) = \dfrac{1}{1+e^{-ay}}$

$$F_Y(y) = \frac{1}{1+e^{-ay}} = x, \quad y = -\frac{1}{a}\ln\left(\frac{1}{x} - 1\right)$$

y = -0.54904, 0.12586, -0.2552, 0.394461, 0.0002, 0.733497, 0.255626, 1.354879, 0.549573, -1.35317, 0.973412, -0.73284, -4.60512, -0.394, -0.9725, -0.12545, -0.54904, …..

What is the corresponding probability density function $f_Y(y)$?

Another Example of RNG

$$s = 2111111111x_{n-4} + 1492x_{n-3} + 1776x_{n-2} + 5115x_{n-1} + c$$

a 64-bit unsigned integer

$$x_n = (s)_{\text{mod } 2^{32}}$$

where $c = \left\lfloor \dfrac{s}{2^{32}} \right\rfloor$ = largest integer less than or equal to $\dfrac{s}{2^{32}}$

The s in calculating the floor above is the previous value (during calculation of x_{n-1}).

Specifications:

- 32-bit integer output
- cycle length is $3 \cdot 10^{47}$

Characteristic Function

- Definition

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{jux} f_X(x)\, dx$$

that is, the characteristic function of a random variable can be viewed as the Fourier transform of its probability density function.

- The pdf is then the inverse Fourier transform of the characteristic function

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jux}\Phi_X(u)\, du$$

$$\frac{d}{du}\Phi(u) = \int_{-\infty}^{\infty}\left(\frac{d}{du}e^{jux}\right) f_X(x)\, dx = \int_{-\infty}^{\infty} jxe^{jux} f_X(x)\, dx$$

$$\left.\frac{d}{du}\Phi(u)\right|_{u=0} = \int_{-\infty}^{\infty} jx\, f_X(x)\, dx = j\overline{X}$$

$$\left.\frac{d^n}{du^n}\Phi(u)\right|_{u=0} = j^n E[X^n] = j^n\,\overline{X^n}$$