ECE 3075A
Random Signals

Lecture 12
Multiple Random Variables & Joint Distributions

School of Electrical and Computer Engineering
Georgia Institute of Technology
Fall, 2003

Characteristic Function

- Definition

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{jux} f_X(x)\, dx$$

that is, the characteristic function of a random variable can be viewed as the Fourier transform of its probability density function.

- The pdf is then the inverse Fourier transform of the characteristic function

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jux}\Phi_X(u)\, du$$

$$\frac{d}{du}\Phi(u) = \int_{-\infty}^{\infty}\left(\frac{d}{du}e^{jux}\right) f_X(x)\, dx = \int_{-\infty}^{\infty} jxe^{jux} f_X(x)\, dx$$

$$\frac{d}{du}\Phi(u)\Big|_{u=0} = \int_{-\infty}^{\infty} jx\, f_X(x)\, dx = j\overline{X} \qquad \frac{d^n}{du^n}\Phi(u)\Big|_{u=0} = j^n E[X^n] = j^n\overline{X^n}$$

Two Random Variables

- As in combined experiments, we are interested in the co-occurrence of separate events.

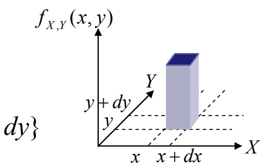
  Try to recall and answer: Do they have to be simultaneous? What is the difference, if any, between tossing a die and a coin together and tossing them separately? What is a combined event in the Cartesian product observation/sample space? What is the definition of **independent** experiments? And, likewise, **dependent** experiment?

- **Joint Probability Distribution Function** of two r.v., X and Y

$$F_{X,Y}(x,y) = \Pr\{X \le x, Y \le y\}$$

1. $0 \le F_{X,Y}(x,y) \le 1, \quad -\infty < x < \infty, -\infty < y < \infty$
2. $F_{X,Y}(-\infty, y) = F_{X,Y}(x,-\infty) = F_{X,Y}(-\infty,-\infty) = 0$
3. $F_{X,Y}(\infty,\infty) = 1$
4. $F_{X,Y}(x,y)$ is a non-decreasing function of either x or y
5. $F_{X,Y}(\infty, y) = F_Y(y), \quad F_{X,Y}(x,\infty) = F_X(x)$

Joint Probability Density Functions



Definition $\quad f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$

$$f_{X,Y}(x,y)\, dx\, dy = \Pr\{x < X \le x+dx, y < Y \le y+dy\}$$

1. $f_{X,Y}(x,y) \ge 0, \quad -\infty < x < \infty, -\infty < y < \infty$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$
3. $F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du$
4. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx \quad \to$ marginals
5. $\Pr\{x_1 < X \le x_2, y_1 < Y \le y_2\} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\, dy\, dx$

Example

The joint density function of X and Y is given by

$$f_{X,Y}(x,y) = u(x)u(y)xe^{-x(y+1)}$$

Find the marginal density functions.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{\infty} u(x)xe^{-x(y+1)}dy$$

$$= u(x)xe^{-x}\int_0^{\infty} e^{-xy}dy = u(x)xe^{-x}\left.\frac{e^{-xy}}{-x}\right|_{y=0}^{\infty} = u(x)e^{-x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{\infty} u(y)xe^{-x(y+1)}dx$$

$$= u(y)e^{-x(y+1)}\left[\frac{x}{-(y+1)} - \frac{1}{(y+1)^2}\right]_{x=0}^{\infty} = \frac{u(y)}{(y+1)^2}$$

note: $xe^{-ax} = \frac{1}{a}\frac{d}{dx}\left(-\frac{1}{a}e^{-ax} - xe^{-ax}\right)$

Expectation of Functions of Two R.V.s

- Similar definition as in single random variable case

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

- When $g(X,Y) = X^n Y^k$

$$E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)\,dx\,dy$$ is called the ***joint moment of*** X ***and*** Y. When n=k=1, it is called correlation. $n+k$ is the order of the moment.

$$E[(X-\bar{X})(Y-\bar{Y})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{X})(y-\bar{Y}) f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy - \bar{X}y - x\bar{Y} + \bar{X}\bar{Y}) f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)\,dx\,dy - \bar{X}\bar{Y} = \overline{XY} - \bar{X}\bar{Y}$$

Joint Central Moments

$$E[(X-\bar{X})^n(Y-\bar{Y})^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{X})^n(y-\bar{Y})^k f_{X,Y}(x,y)\,dx\,dy = \mu_{nk}$$

$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy = R_{XY}$ is the correlation between X and Y

If $R_{XY} = E[XY] = E[X]E[Y]$, then X and Y are said to be uncorrelated.

If $R_{XY} = 0$, then X and Y are said to be orthogonal.

μ_{11} is called covariance.

$$\rho = E\left\{\left[\frac{X-\bar{X}}{\sigma_X}\right]\left[\frac{Y-\bar{Y}}{\sigma_Y}\right]\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x-\bar{X}}{\sigma_X}\frac{y-\bar{Y}}{\sigma_Y} f_{X,Y}(x,y)\,dx\,dy$$

ρ is called the **correlation coefficient** or **normalized covariance**, which expresses the degree of correlation between the two r.v.s without regard to the magnitude of either one r.v.

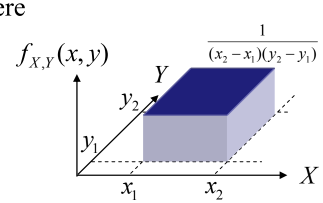
Two Uniform Random Variables

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{(x_2-x_1)(y_2-y_1)}, & x_1 < x \le x_2,\ y_1 < y \le y_2 \\ 0, & \text{elsewhere} \end{cases}$$

Marginals

$$f_X(x) = \int_{y_1}^{y_2} \frac{1}{(x_2-x_1)(y_2-y_1)}dy = \left.\frac{y}{(x_2-x_1)(y_2-y_1)}\right|_{y_1}^{y_2} = \frac{1}{x_2-x_1}$$

$$f_Y(y) = \int_{x_1}^{x_2} \frac{1}{(x_2-x_1)(y_2-y_1)}dx = \left.\frac{x}{(x_2-x_1)(y_2-y_1)}\right|_{x_1}^{x_2} = \frac{1}{y_2-y_1}$$



$$E[XY] = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \frac{xy}{(x_2-x_1)(y_2-y_1)}dx\,dy = \frac{1}{(x_2-x_1)(y_2-y_1)}\left[\left.\frac{x^2}{2}\right|_{x_1}^{x_2}\right]\left[\left.\frac{y^2}{2}\right|_{y_1}^{y_2}\right] = \frac{(x_1+x_2)(y_1+y_2)}{4}$$

$$F_{X,Y}(x,y) = \int_{x_1}^{x}\int_{y_1}^{y} \frac{1}{(x_2-x_1)(y_2-y_1)}dv\,du = \frac{(x-x_1)(y-y_1)}{(x_2-x_1)(y_2-y_1)}$$