ECE 3075A
Random Signals

Lecture 14
Distributions of Functions of Several Random Variables

School of Electrical and Computer Engineering
Georgia Institute of Technology
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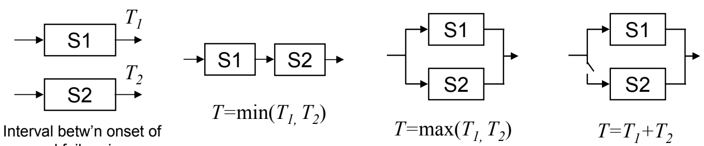
Probability Distributions of Functions of Several Random Variables

- Observation data usually can be modeled by certain distributions – we have an extensive set of distribution functions to work with (from normal distribution to mixture).
- We study the joint distribution of several random events to understand and make use of the statistical relationship among them. For example, the noise level and the bit error rate in digital communication.
- Yet often we are interested in functions of these random events, to construct new knowledge or for new use that is not obvious from the random events themselves.



Probability Distributions of Functions of Several Random Variables

- The probability space is defined on a hyperspace that contains the random variables. *(try to make sure you understand the difference between Y=g(X), a function of a r.v., and Y, a random variable itself.)*
- Function $Y = g(X_1, X_2, \cdots, X_N)$

$$F_Y(y) = \Pr(Y \le y) = \Pr[g(X_1, X_2, \cdots, X_N) \le y]$$
$$= \Pr\left(\{\xi : g(x_1 = X_1(\xi), x_2 = X_2(\xi), \cdots, x_N = X_N(\xi)) \le y\}\right)$$

$$F_Y(y) = \Pr[g(X_1, X_2, \cdots, X_N) \le y] = \int\cdots\int_{\{g(x_1,x_2,\cdots,x_N)\le y\}} f_{X_1,X_2,\cdots,X_N}(x_1, x_2, \cdots, x_N)dx_1 dx_2 \cdots dx_N$$

⟹ Direct integration

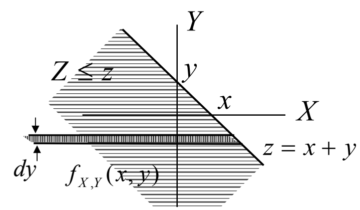
- Another method is through transformation of variables.

Sum of Two Random Variables

$$Z = X + Y$$

At any y, the small stripe has a probability mass of

$$dy\int_{-\infty}^{z-y} f_{X,Y}(x, y)dx$$



Thus, the shaded area which is $\Pr\{Z \le z\}$ can be obtained by

$$\Pr\{Z \le z\} = F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{X,Y}(x, y)\,dx\,dy$$

If X and Y are independent, $F_Z(z) = \int_{-\infty}^{\infty} f_Y(y)\int_{-\infty}^{z-y} f_X(x)\,dx\,dy$

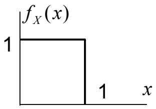
And, $f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y)\,dy$ Use Leibniz's rule

The pdf of the sum of two statistically independent random variables is the convolution of their individual pdf's.

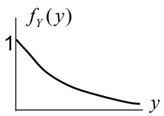
Sum of Random Variables - Example

The pdf of X and Y, which are independent, are

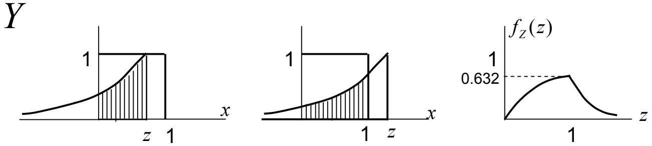
$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$



$$f_Y(y) = \begin{cases} e^{-y}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$



$$Z = X + Y$$



$0 < z \le 1$

$$f_Z(z) = \int_0^1 e^{-(z-x)} u(z-x)dx = \int_0^z e^{-(z-x)}dx = e^{-z}[e^x]_{x=0}^z = 1 - e^{-z}$$

$1 < z < \infty$

$$f_Z(z) = \int_0^1 e^{-(z-x)} u(z-x)dx = \int_0^1 e^{-(z-x)}dx = e^{-z}[e^x]_{x=0}^1 = e^{-z}(e-1)$$

Probability Density Functions of Two R.V.s

$$Z = \min(X, Y)$$

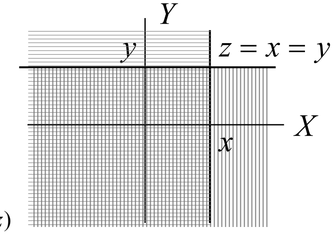
$\{Z \le z\} = \{X \le z\} \cup \{Y \le z\}$

$\Pr\{Z \le z\} = \Pr\{X \le z\} + \Pr\{Y \le z\}$
$\qquad - \Pr\{X \le z, Y \le z\}$

$F_Z(z) = F_X(z) + F_Y(z) - F_{X,Y}(z,z)$

If X and Y are independent,

$f_Z(z) = f_X(z) + f_Y(z) - f_X(z)F_Y(z) - F_X(z)f_Y(z)$
$= f_X(z)[1 - F_Y(z)] + f_Y(z)[1 - F_X(z)]$



$$Z = \max(X, Y)$$

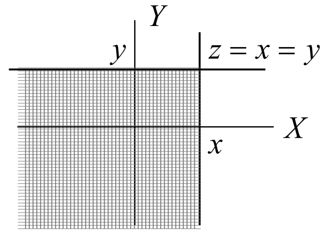
$\Pr\{Z \le z\} = \Pr\{X \le z, Y \le z\}$

If X and Y are independent,

$\Pr\{Z \le z\} = \Pr\{X \le z\}\Pr\{Y \le z\}$

$\therefore\ F_Z(z) = F_X(z)F_Y(z)$ and

$f_Z(z) = f_X(z)F_Y(z) + f_Y(z)F_X(z)$



Probability Density Functions of Two R.V.s

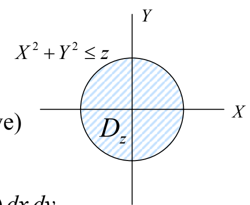
$$Z = X^2 + Y^2$$

If $z < 0$,

$F_Z(z) = 0$ (because, $X^2 + Y^2$ can never be negative)

If $z > 0$,

$$F_Z(z) = \Pr\{Z \le z\} = \Pr\{X^2 + Y^2 \le z\} = \iint_{x^2+y^2 \le z} f_{X,Y}(x,y)\,dx\,dy$$



Example:

Given $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$, find pdf for $Z = X^2 + Y^2$.

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then, $dxdy = rdrd\theta$.

$$F_Z(z) = \int_0^{2\pi}\int_0^{\sqrt{z}} \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} rdrd\theta = \int_0^{\sqrt{z}} \frac{2\pi r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} dr = 1 - \exp\left(-\frac{z}{2\sigma^2}\right), \quad z > 0$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{2\sigma^2}\exp\left(-\frac{z}{2\sigma^2}\right)u(z)$$

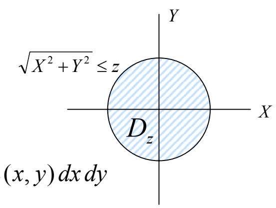
Probability Density Functions of Two R.V.s

$$Z = \sqrt{X^2 + Y^2}$$

If $z < 0$, $\quad F_Z(z) = 0$

If $z > 0$,

$$F_Z(z) = \Pr\{Z \le z\} = \Pr\{\sqrt{X^2 + Y^2} \le z\} = \iint_{\sqrt{x^2+y^2} \le z} f_{X,Y}(x,y)\,dx\,dy$$



Example:

Given $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$, find pdf for $Z = \sqrt{X^2 + Y^2}$.

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then, $dxdy = rdrd\theta$.

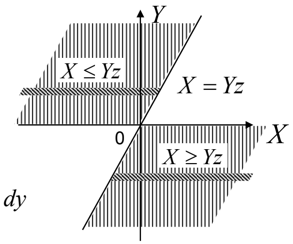
$$F_Z(z) = \int_0^{2\pi}\int_0^{z} \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} rdrd\theta = \int_0^{z} \frac{2\pi r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} dr = 1 - \exp\left(-\frac{z^2}{2\sigma^2}\right), \quad z > 0$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{z}{\sigma^2}\exp\left(-\frac{z^2}{2\sigma^2}\right)u(z)$$

Probability Density Functions of Two R.V.s

$Z = X / Y$

Given z, the function $x = yz$ is a line going through the origin.



$$F_Z(z) = \Pr\{Z \le z\} = \Pr\left\{\frac{X}{Y} \le z\right\} = \iint_{\text{shaded area}} f_{X,Y}(x,y)\,dx\,dy$$

because if $y > 0$, then $x \le yz$; if $y < 0$, then $x \ge yz$

$$F_Z(z) = \int_0^{\infty}\int_{-\infty}^{yz} f_{X,Y}(x,y)\,dx\,dy + \int_{-\infty}^{0}\int_{yz}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$f_Z(z) = \int_0^{\infty} y\, f_{X,Y}(yz,y)\,dy - \int_{-\infty}^{0} y f_{X,Y}(yz,y)\,dy = \int_{-\infty}^{\infty} |y| f_{X,Y}(yz,y)\,dy$$