ECE 3075A
Random Signals

Lecture 24

Frequently Encountered Random Processes - II

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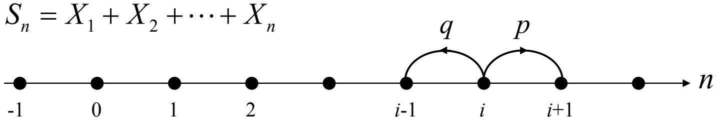
Random Walks

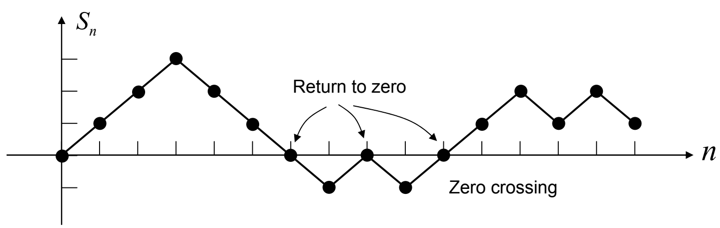
- $X_1, X_2, \cdots, X_n, \cdots$ are a sequence of independent random variables that assume values +1 and –1 with probability p and q=1-p, respectively. (The value of 1 is arbitrary subject to scaling.)
- Let $S_n = X_1 + X_2 + \cdots + X_n$ be the partial sum with $S_0 = 0$. The partial sum represents the accumulated positive or negative results at the end of nth trial.
- Many real life phenomena can be modeled by a random walk process.
  - The stock value variations of a particular stock;
  - The number of sign changes (or zero crossing, or level crossing) of a signal
  - The motion of gas molecules in a diffusion process
- If p=q, it is called a symmetric random walk; otherwise, an asymmetric random walk.

Random Walks

$S_n = X_1 + X_2 + \cdots + X_n$



Integer state representation: S_n is the state the system is in.



Time function representation: S_n is a function of "time" index.

Random Walks

Let the event { $S_n = r$ } be the event "at time n the system is at point r," and $p_{n,r} = \Pr\{S_n = r\}$

$$p_{n,r} = \Pr\{S_n = r\} = \binom{n}{k} p^k q^{n-k}$$

where k is the number of +1's in n trials and n-k the number of –1's, and the net gain, which is r, is

$$r = k - (n-k) = 2k - n \quad \text{or} \quad k = (n+r)/2$$

Thus,

$$p_{n,r} = \binom{n}{(n+r)/2} p^{(n+r)/2} q^{(n-r)/2}$$

where $(n+r)/2$ is an integer between 0 and n inclusive; that is, n and r must be both odd or both even at the same time.

Random Walks - Examples

What is the probability that $S_n = r = n$?

$p_{n,n} = \Pr\{S_n = n\} = \Pr\{\text{walk in the same positive direction for } n \text{ times}\}$

$$p_{n,n} = \Pr\{S_n = n\} = \binom{n}{n} p^k q^{n-k} = p^n$$

What is the probability that $S_n = r = 0$?

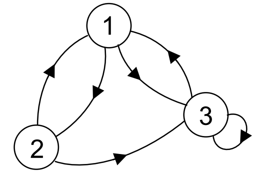
$$p_{n,0} = \binom{n}{n/2} p^{n/2} q^{n/2} = \binom{n}{n/2} p^{n/2} (1-p)^{n/2}$$

The polynomial coefficient indicates that there are $\binom{n}{n/2} = \frac{n!}{(n/2)!(n/2)!}$ paths that contain equal number of upward and downward steps.

What is the probability that a random walk process will reach a place within 2 steps of the peak, given that n steps have taken place?

Markov Chain and Processes

- Markov Process Taxonomy
  - By state space
    - Discrete: Markov Chain
    - Continuous: Markov process
  - By time of transition
    - Continuous time
    - Discrete time



Discrete state
At state 1, $X=b_1$
At state 2, $X=b_2$
At state 3, $X=b_3$

$X(t) = b_i, \quad i = 1, 2, \cdots, N$ depending on the state the system is in.

Markov Property:

$$\Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \cdots, X(t_1) = x_1\}$$
$$= \Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\} \quad \text{where } t_1 < t_2 < \cdots < t_n$$

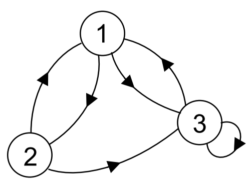
Finite State Markov Chain

- Markov Chain

$$\Pr\{X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \cdots, X(t_1) = x_1\}$$
$$= \Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \cdots, X(t_1) = x_1\}$$
$$\bullet \Pr\{X(t_{n-1}) = x_{n-1} \mid X(t_{n-2}) = x_{n-2}, \cdots, X(t_1) = x_1\} \bullet \cdots \bullet \Pr\{X(t_2) = x_2 \mid X(t_1) = x_1\} \bullet \Pr\{X(t_1) = x_1\}$$
$$= \Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\} \Pr\{X(t_{n-1}) = x_{n-1} \mid X(t_{n-2}) = x_{n-2}\} \bullet \cdots \bullet \Pr\{X(t_2) = x_2 \mid X(t_1) = x_1\} \bullet \Pr\{X(t_1) = x_1\}$$

- An N-state (1st order) Markov chain is characterized by an initial state probability array, $\{\Pr[X(t_1)]\}_{\text{all states}}$, and a transition probability matrix $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \cdots & & \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \qquad \sum_{j=1}^{N} a_{ij} = 1$$



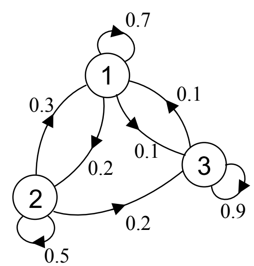
where $a_{ij} = \Pr\{X(t_n) = b_j \mid X(t_{n-1}) = b_i\}$ where $t_{n+1} < t_n$.

Markov Chain - Example

- A finite-state Markov Chain is governed by the following parameters:

The initial probability vector: $[0.7 \quad 0.2 \quad 0.1]^t$

The state transition matrix: $\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.0 & 0.9 \end{bmatrix}$

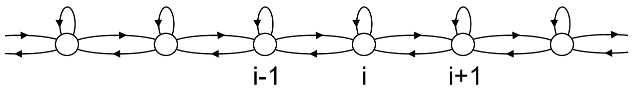


What is the probability that the system goes through the following sequence of states: (1, 3, 3, 1, 3, 1, 2, 1, 1, 3)

$$\Pr\{(1,3,3,1,3,1,2,1,1,3)\}$$
$$= 0.7 \times 0.1 \times 0.9 \times 0.1 \times 0.1 \times 0.1 \times 0.2 \times 0.3 \times 0.7 \times 0.1 = 2.646 \times 10^{-7}$$

Other Random Processes

- Birth-Death Processes: special case of Markov Process in that the **state** can only move up (one birth) or down (one death) by one or stay the same (no birth or death); however, the value associated with a state can be arbitrary (unlike random walk in which the random variable changes **value** by the same amount, say 1, up or down).



- Renewal Processes: related to random walk but with interest in counting the transitions that takes place as a function of time.