ECE 3075A
Random Signals

Lecture 26
Autocorrelation Functions of Random Binary Processes

School of Electrical and Computer Engineering
Georgia Institute of Technology
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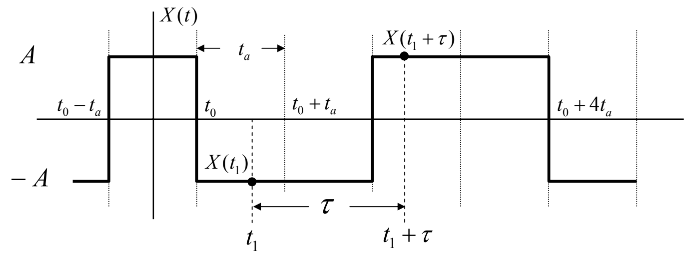
Autocorrelation of a Binary Process



- A discrete (equi-probable at $\pm A$), stationary, zero-mean process.
- State change clocked at t_a interval with arbitrary starting time, t_0; that is, t_0 is considered a random variable uniformly distributed over (0, t_a).
- $X(t)$ in one interval is statistically independent from $X(t)$ in another interval.
- The process is very common in data communications and digital computers.

Autocorrelation of a Binary Process

$X(t_1)$ and $X(t_2)$ are independent if $|t_1 - t_2| = |\tau| > t_a$.

Therefore, due to the fact that the process is stationary, zero-mean,

$$R_X(\tau) = E[X(t)X(t+\tau)] = E[X(t)]E[X(t+\tau)] = 0, \quad |\tau| > t_a$$

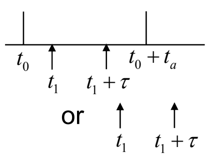
When $|\tau| < t_a$, t_1 and $t_2 = t_1 + \tau$ may or may not be in the same interval, depending on the value of t_0.

$\Pr\{t_1$ and $t_1 + \tau$, $\tau > 0$, are in the same interval $\}$

$$= \Pr\{t_1 + \tau - t_a < t_0 \le t_1\} = \frac{1}{t_a}[t_1 - (t_1 + \tau - t_a)] = \frac{t_a - \tau}{t_a}$$

$\Pr\{t_1$ and $t_1 + \tau$, $\tau < 0$, are in the same interval $\}$

$$= \Pr\{t_1 - t_a < t_0 \le t_1 + \tau\} = \frac{1}{t_a}[t_1 + \tau - (t_1 - t_a)] = \frac{t_a + \tau}{t_a}$$



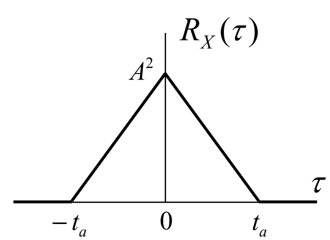
$$\Pr\{t_1 \text{ and } t_1 + \tau \text{ are in the same interval}\} = \frac{t_a - |\tau|}{t_a}$$

Autocorrelation of a Binary Process

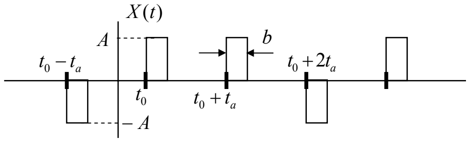
When t_1 and t_2 are in the same interval, the product of X_1 and X_2 is always A^2; when they are not, X_1 and X_2 are independent with zero mean and thus zero correlation. Hence,

$$R_X(\tau) = \begin{cases} A^2\left[\dfrac{t_a - |\tau|}{t_a}\right] = A^2\left[1 - \dfrac{|\tau|}{t_a}\right], & |\tau| \le t_a \\ 0, & |\tau| > t_a \end{cases}$$

Remarks:
- When the two time instances are close to each other, the two corresponding r.v.s are likely to have the same value;
- When they are apart far enough, it is equally probable that they'll have the same value as they'll have the opposite value;
- At $\tau = 0$, the autocorrelation is the same as the mean square value, representing the power of the signal.



Example 6-2.2



The process is the same as the binary process previously discussed except that it now does not have a full duty cycle but only b/t_a .

The product of $X(t_1)$ and $X(t_2)$ is zero if $|t_1 - t_2| = |\tau| > b$.

When $|\tau| < b$, t_1 and $t_2 = t_1 + \tau$ may or may not be in the same interval.

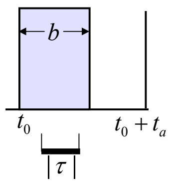
The range that the center of the time-bar ($|\tau|$ wide) can be in for it to be totally in the shaded area is $\left(t_0 + \frac{|\tau|}{2}, t_0 + b - \frac{|\tau|}{2}\right)$

This range has a width of $b - |\tau|$.

Therefore,

$$\Pr\{t_1 \text{ and } t_1 + \tau \text{ are in the same active duty interval}\} = \frac{b - |\tau|}{t_a}$$

$$R_X(\tau) = \begin{cases} A^2\left[\frac{b - |\tau|}{t_a}\right] = A^2 \frac{b}{t_a}\left[1 - \frac{|\tau|}{b}\right], & |\tau| \le b \\ 0, & |\tau| > b \end{cases}$$

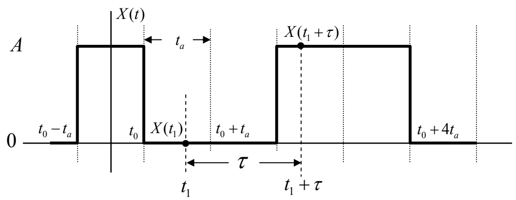


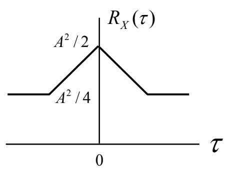
b/t_a is called the duty cycle.

Other Examples of Autocorrelation Functions

1. Binary process with uniformly spaced switching intervals – see previous discussion.
2. Binary process with uniformly spaced switching intervals and non-zero mean.
3. Binary process with randomly spaced switching times – the telegraph process.
4. Bandpass filtered signals – discussion will take place with introduction of power spectrum.

Binary Process with Non-zero Mean





$X(t) = \frac{A}{2} + \frac{1}{2}X'(t)$ where $X'(t)$ is the binary process previously discussed.

$$R_X(\tau) = E[X(t)X(t+\tau)] = \frac{A^2}{4} + \frac{A}{4}E[X'(t)] + \frac{A}{4}E[X'(t+\tau)] + \frac{1}{4}E[X'(t)X'(t+\tau)]$$

$$= \frac{A^2}{4} + \frac{1}{4}R_{X'}(\tau) = \frac{A^2}{4} + \frac{1}{4}R_{X'}(\tau)$$

$$R_{X'}(\tau) = \begin{cases} A^2\left[\frac{t_a - |\tau|}{t_a}\right] = A^2\left[1 - \frac{|\tau|}{t_a}\right], & |\tau| \le t_a \\ 0, & |\tau| > t_a \end{cases}$$

Therefore,

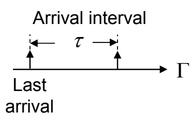
$$R_X(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4}\left[1 - \frac{|\tau|}{t_a}\right], & |\tau| \le t_a \\ \frac{A^2}{4}, & |\tau| > t_a \end{cases}$$

Arrival Interval of A Poisson Process

- Probability of arrival interval, τ, as a random variable is the same as the probability that there is no arrival during that interval.

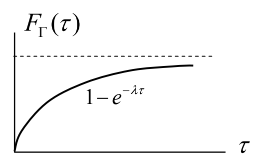
$$\Pr\{X(\tau) = 0\} = \frac{(\lambda\tau)^0}{0!}e^{-\lambda\tau} = e^{-\lambda\tau}$$

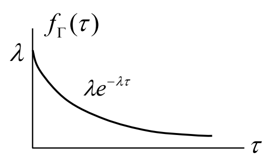
$$F_\Gamma(\tau) = \Pr\{\text{arrival interval} \le \tau\}$$
$$= 1 - \Pr\{\text{no arrival in } (0, \tau)\}$$
$$= 1 - \Pr\{X(\tau) = 0\} = 1 - e^{-\lambda\tau}, \quad \tau \ge 0$$



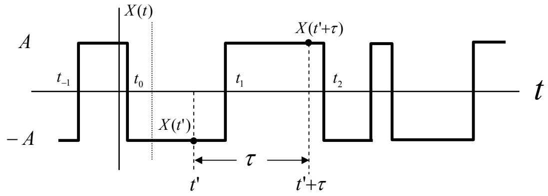
$$f_\Gamma(\tau) = \frac{d}{d\tau}F_\Gamma(\tau) = \lambda e^{-\lambda\tau}, \quad \tau \ge 0$$

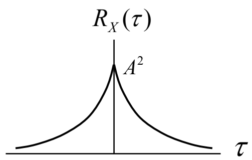
An exponential distribution!





Binary Process with Random Switching Times





- The switching times $\{t_i\}_{i=-\infty}^{\infty}$ occur as Poisson arrivals – a point process.
- When τ is less than the switching interval, the correlation between $X(t')$ and $X(t'+\tau)$ has a value A^2.

$$\Pr\{t' \text{ and } t'+\tau \text{ fall within a switching interval}\}$$
$$= \Pr\{\text{no switching or arrival in } (0,|\tau|)\} = \exp(-\lambda|\tau|)$$

Therefore, $R_X(\tau) = A^2 \exp(-\lambda|\tau|)$ — λ is the rate of switching (average number of switching per unit time).