

**ECE 3075A**  
**Random Signals**

Lecture 27  
**Crosscorrelation Functions**

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## Crosscorrelation Functions

- The correlation function between two different random processes is called a crosscorrelation function.
- The crosscorrelation function measures how “coherently” two processes behave together, at various points in time.
- Crosscorrelation functions are very much of interest in system analysis; for example, in studying the relationship between the input and the output of an electronic system, the relationship between the interest rate and various market indices, etc.

## Jointly W-S Stationary Processes

- A process  $X(t)$  is wide sense stationary if  
 $E[X(t)] = \bar{X} = \text{constant}$  (independent of time), and  
 $E[X(t)X(t+\tau)] = R_{XX}(\tau)$ , a function of only the time difference.
- Two processes  $X(t)$  and  $Y(t)$  are jointly wide sense stationary if
  1. Both  $X(t)$  and  $Y(t)$  are wide sense stationary, and
  2.  $R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$ ,  
a function of only the time difference.

Recall the original definition:

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1 y_2 f_{X_1 Y_2}(x_1, y_2) dy_2$$

where we have used the notation  $X_1 = X(t_1)$ ,  $Y_2 = Y(t_2)$

## Crosscorrelation of W-S Stationary Processes

- $X(t)$  and  $Y(t)$  are jointly stationary in wide sense.
- $X_1$  and  $Y_2$  are two random variables,  
 $X_1 = X(t_1)$  and  $Y_2 = Y(t_1 + \tau)$
- The crosscorrelation function is defined as  
$$R_{XY}(\tau) = E[X_1 Y_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} x_1 y_2 f_{X_1 Y_2}(x_1, y_2) dy_2$$
- The temporal order is of significance:  
$$R_{YX}(\tau) = E[Y_1 X_2] = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} y_1 x_2 f_{Y_1 X_2}(y_1, x_2) dx_2$$
  
In the above,  $Y_1 = Y(t_1)$  and  $X_2 = X(t_1 + \tau)$ .

Let  $t_1 = t_2 - \tau$ , it then follows that  $t_2 = t_1 + \tau$  and  $R_{XY}(\tau) = R_{YX}(-\tau)$ .

## Time Crosscorrelation Functions

• Define 
$$\mathbf{R}_{XY}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)y(t+\tau) dt$$

$$\mathbf{R}_{YX}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T y(t)x(t+\tau) dt$$

- If the processes are jointly ergodic,

$$\mathbf{R}_{XY}(\tau) = R_{XY}(\tau), \text{ and } \mathbf{R}_{YX}(\tau) = R_{YX}(\tau)$$

Example:

Two jointly random processes are of the form

$$X(t) = 2 \cos(5t + \Theta) \text{ and } Y(t) = 10 \sin(5t + \Theta)$$

where  $\Theta$  is a random variable uniformly distributed in  $(0, 2\pi)$ . Find the crosscorrelation of the two processes

$$\begin{aligned} R_{XY}(\tau) &= E[2 \cos(5t + \Theta) 10 \sin(5t + 5\tau + \Theta)] \\ &= 10 \sin(5\tau) + E[10 \sin(10t + 5\tau + 2\Theta)] = 10 \sin(5\tau) \end{aligned}$$

## Properties of Crosscorrelation Functions

- $R_{XY}(0)$  and  $R_{YX}(0)$  only measures the correlation of the two processes at synchronous points.

- $R_{XY}(0) = R_{YX}(0)$ ;  $R_{XY}(\tau) = R_{YX}(-\tau)$

- $|R_{XY}(\tau)| \leq [R_X(0)R_Y(0)]^{1/2}$

- The maximum of a crosscorrelation function can occur anywhere, not necessarily at  $\tau = 0$ .

- If the two random processes are independent,

$$R_{XY}(\tau) = E[X_1 Y_2] = E[X_1]E[Y_2] = \bar{X}_1 \bar{Y}_2 = R_{YX}(\tau)$$

And if any of the process has zero mean, the crosscorrelation function vanishes everywhere.

## Applications of Crosscorrelation Functions

$$Z(t) = X(t) \pm Y(t)$$

$$Z_1 = Z(t_1) = X(t_1) \pm Y(t_1) = X_1 \pm Y_1$$

$$Z_2 = Z(t_1 + \tau) = X(t_1 + \tau) \pm Y(t_1 + \tau) = X_2 \pm Y_2$$

$$R_Z(\tau) = E[Z_1 Z_2] = E[(X_1 \pm Y_1)(X_2 \pm Y_2)]$$

$$= E[X_1 X_2 + Y_1 Y_2 \pm X_1 Y_2 \pm Y_1 X_2]$$

$$= R_X(\tau) + R_Y(\tau) \pm R_{XY}(\tau) \pm R_{YX}(\tau)$$

The autocorrelation function of the sum is the sum of all the autocorrelation functions plus the sum of all the crosscorrelation functions.

If the two random processes are statistically independent and one of them has zero mean, the cross terms vanish, resulting in "the autocorrelation function of the sum is the sum of the autocorrelation functions."

$$R_{XY}(\tau) = E[X_1 Y_2] = E[X_1]E[Y_2] = 0 \text{ if } \bar{X}_1 = 0 \text{ or } \bar{Y}_2 = 0.$$

## Sinusoid Plus Noise

Let  $X(t) = A \cos(\omega t + \Theta)$  where  $\Theta$  is a random variable uniformly distributed over  $(0, 2\pi)$ .

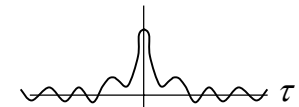
$$R_X(\tau) = \frac{1}{2} A^2 \cos \omega \tau$$

Let  $V(t)$  be a zero mean noise process, statistically independent of  $X(t)$ , the signal, with autocorrelation function:

$$R_V(\tau) = B^2 e^{-\alpha|\tau|}$$

The observed process is  $Z(t) = A \cos(\omega t + \Theta) + V(t)$  which has an autocorrelation function:

$$R_Z(\tau) = \frac{1}{2} A^2 \cos \omega \tau + B^2 e^{-\alpha|\tau|}$$



Note that  $R_V(\tau) = B^2 e^{-\alpha|\tau|} \rightarrow 0$ , as  $\tau \rightarrow \infty$ .

It is thus possible to recover a sinusoid from noise contamination as long as we measure the autocorrelation at sufficiently long time lags.

## Use of Crosscorrelation - Example

A radar sends out a signal  $X(t)$  which is returned from a target, after a round-trip delay of  $\tau_1$ . The receiver receives  $Y(t)$ , which includes the delayed and attenuated signal  $X(t - \tau_1)$  and the noise  $V(t)$  :

$$Y(t) = aX(t - \tau_1) + V(t) \quad \text{where } a < 1$$

We use the crosscorrelation function between  $X$  and  $Y$  to detect the presence of the returned signal in  $Y$ .

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t + \tau)] \\ &= E[aX(t)X(t + \tau - \tau_1) + X(t)V(t + \tau)] \\ &= aR_X(\tau - \tau_1) + R_{XV}(\tau) = aR_X(\tau - \tau_1) \end{aligned}$$

because  $X$  and  $V$  are statistically independent and  $V$  has zero mean.

$R_{XY}(\tau) = aR_X(\tau - \tau_1)$  attains maximum at  $\tau = \tau_1$ , from which the distance to the target can be calculated.

## Example – 6-8.1

A random process has the form  $X(t)=A$  in which  $A$  is a random variable with mean value 5 and variance 10. This random process can be observed only in the presence of independent noise  $V(t)$  having an auto correlation function of

$$R_V(\tau) = 10 \exp(-2|\tau|)$$

1. Find the autocorrelation function of the sum of these two processes;
2. If the autocorrelation function of the sum is observed, find the value of  $\tau$  at which the autocorrelation function is within 1% of its value at  $\tau = \infty$ .

$$Y(t) = X(t) + V(t)$$

$$R_Y(\tau) = E[Y(t)Y(t + \tau)]$$

$$\begin{aligned} &= E[X(t)X(t + \tau)] + E[X(t)V(t + \tau)] + E[V(t)X(t + \tau)] + E[V(t)V(t + \tau)] \\ &= E[A^2] + R_V(\tau) = \sigma_A^2 + \bar{A}^2 + R_V(\tau) = 35 + 10 \exp(-2|\tau|) \end{aligned}$$

$$R_Y(\tau) = 35 + 10 \exp(-2|\tau|), \quad R_Y(\infty) = 35$$

$$R_Y(\tau) = 35 + 10 \exp(-2|\tau|) = 35.35 \Rightarrow \exp(-2|\tau|) = 0.035$$

Therefore, at  $\tau = \pm(-0.5 \ln 0.035) = \pm 1.6762$   $R_Y(\tau)$  is within 1% of  $R_Y(\infty)$ .

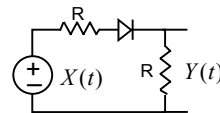
## Example – 6-8.2

A random binary process (see Section 6-2) has amplitude of  $\pm 12$  and  $t_a = 0.01$ . It is applied to the half-wave rectifier circuit as shown on the right.

Find the autocorrelation of the output,  $R_Y(\tau)$

Find the crosscorrelation function  $R_{XY}(\tau)$

Find the crosscorrelation function  $R_{YX}(\tau)$ .



The amplitude of  $Y(t)$  is either 0 (when  $X(t)$  is -12) or half of  $X(t)$  when it is +12, i.e. +6. The mean of  $Y(t)$  is thus 3.

$Y(t) = \bar{Y} + W(t)$  where  $W(t)$  is a zero - mean binary process with amplitude  $\pm 3$ .

$$R_Y(\tau) = \bar{Y}^2 + R_W(\tau) = \begin{cases} 9 + 9 \left(1 - \frac{|\tau|}{0.01}\right), & |\tau| \leq 0.01 \\ 9, & |\tau| > 0.01 \end{cases}$$

For crosscorrelation:

If  $|\tau| > t_a$ ,  $X(t)$  and  $Y(t)$  are independent and  $X(t)$  has zero mean, thus  $R_{XY}(\tau) = 0$ .

If  $|\tau| \leq t_a$ ,  $Y(t)$  is half (= +6) of  $X(t)$  (= +12) half of the time; the other half of time,

$X(t)$  is negative and  $Y(t)$  is 0. Thus  $R_{XY}(\tau) = \frac{1}{2} \times 12 \times 6 \times \left(1 - \frac{|\tau|}{0.01}\right) = 36 \left(1 - \frac{|\tau|}{0.01}\right)$ .