ECE 3075A
Random Signals

Lecture 28

Discrete-time Random Processes

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Discrete-Time Sequences as R. P.

$X(t)$ is observed at equally spaced time instants. Let

$X_1 = X(\Delta t), X_2 = X(2\Delta t), \cdots, X_N = X(N\Delta t)$, then

$$\hat{\bar{X}} = \frac{1}{N}\sum_{i=1}^{N} X_i \text{ which is a discrete-time version of } \hat{\bar{X}} = \frac{1}{T}\int_{t=0}^{T} X(t)\,dt.$$

$$E[\hat{\bar{X}}] = E\left[\frac{1}{N}\sum_{i=1}^{N} X_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[X_i] = \frac{1}{N}\sum_{i=1}^{N} \bar{X} = \bar{X}$$

$$E[(\hat{\bar{X}})^2] = E\left[\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j\right] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E[X_i X_j]$$

If these time-sampled random variables are statistically independent,

$$E[X_i X_j] = \overline{X^2} \text{ if } i = j; \ = (\bar{X})^2 \text{ if } i \neq j.$$

$$\text{Thus, } E[(\hat{\bar{X}})^2] = \frac{1}{N^2}[N\overline{X^2} + (N^2 - N)\bar{X}^2]$$

Discrete-Time Processes

$$E[(\hat{\bar{X}})^2] = \frac{1}{N^2}[N\overline{X^2} + (N^2 - N)\bar{X}^2] = \frac{\overline{X^2}}{N} + \left(1 - \frac{1}{N}\right)\bar{X}^2 = \frac{\sigma_X^2}{N} + \bar{X}^2$$

$$\text{var}(\hat{\bar{X}}) = E[(\hat{\bar{X}})^2] - \bar{X}^2 = \frac{\sigma_X^2}{N}$$

The variance is $1/N$ of the variance of the process. Therefore, the more data used in time average for estimating the mean of the process, the better the result. This fact is similar to the sample mean when the statistics of a population was discussed.

Similarly, the sample variance is a biased estimate of the variance of the process. We eliminate the bias by using the following as an unbiased estimate of the variance of the process:

$$\hat{\sigma}_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \hat{\bar{X}})^2 = \frac{1}{N-1}\sum_{i=1}^{N} X_i^2 - \frac{N}{N-1}(\hat{\bar{X}})^2$$

Autocorrelation of Discrete-Time Sequences

Use of a "partial time autocorrelation function" as an estimate of the autocorrelation function of the process:

$$\hat{R}_X(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} x(t)x(t+\tau)\,dt$$

Implemented in discrete time,

$$\hat{R}_X(n\Delta t) = \frac{1}{N-n+1}\sum_{k=0}^{N-n} x_k x_{k+n} \qquad n = 0, 1, 2, \cdots, M, \ M << N$$

Quality of estimate: Unbiased

$$E[\hat{R}_X(n\Delta t)] = E\left[\frac{1}{N-n+1}\sum_{k=0}^{N-n} X_k X_{k+n}\right] = \frac{1}{N-n+1}\sum_{k=0}^{N-n} E[X_k X_{k+n}] = \frac{1}{N-n+1}\sum_{k=0}^{N-n} R_X(n\Delta t) = R_X(n\Delta t)$$

However, for a smaller mean-square error, often used is

$$\hat{R}_X(n\Delta t) = \frac{1}{N+1}\sum_{k=0}^{N-n} x_k x_{k+n} \qquad n = 0, 1, 2, \cdots, M, \ M << N$$

Autocorrelation Matrices

Use vector notations,

$$\mathbf{X} = [X(t_1) \quad X(t_2) \quad \cdots \quad X(t_N)]^t = [X_1 \quad X_2 \quad \cdots \quad X_N]^t$$

Then,

$$\mathbf{R_X} = E[\mathbf{XX}^t] = \begin{bmatrix} E[X(t_1)X(t_1)] & E[X(t_1)X(t_2)] & \cdots & E[X(t_1)X(t_N)] \\ E[X(t_2)X(t_1)] & E[X(t_2)X(t_2)] & \cdots & E[X(t_2)X(t_N)] \\ \cdots & \cdots & \cdots & \cdots \\ E[X(t_N)X(t_1)] & E[X(t_N)X(t_2)] & \cdots & E[X(t_N)X(t_N)] \end{bmatrix}$$

$$= \begin{bmatrix} R_X(t_1,t_1) & R_X(t_1,t_2) & \cdots & R_X(t_1,t_N) \\ R_X(t_2,t_1) & R_X(t_2,t_2) & \cdots & R_X(t_2,t_N) \\ \cdots & \cdots & \cdots & \cdots \\ R_X(t_N,t_1) & R_X(t_N,t_2) & \cdots & R_X(t_N,t_N) \end{bmatrix}$$

Often we are dealing with wide sense stationary processes sampled at discrete times. Then,

$$\mathbf{R_X} = \begin{bmatrix} R_X[0] & R_X[\Delta t] & \cdots & R_X[(N-1)\Delta t] \\ R_X[\Delta t] & R_X[0] & \cdots & R_X[(N-2)\Delta t] \\ \cdots & \cdots & \cdots & \cdots \\ R_X[(N-1)\Delta t] & R_X[(N-2)\Delta t] & \cdots & R_X[0] \end{bmatrix}$$

Covariance Matrices

Also we often are dealing with normal (Gaussian) processes. (Recall joint distributions of Gaussian random variables.) The covariance matrix is a more direct quantity in the definition of joint distributions than the autocorrelation matrix. $\mathbf{C}$ is the same as $\mathbf{\Lambda}$ in the text.

$$\mathbf{C}_X = E[(\mathbf{X}-\overline{\mathbf{X}})(\mathbf{X}-\overline{\mathbf{X}})^t] = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1N}\sigma_1\sigma_N \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{2N}\sigma_2\sigma_N \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{N1}\sigma_N\sigma_1 & \rho_{N2}\sigma_N\sigma_2 & \cdots & \sigma_N^2 \end{bmatrix}$$

$$\mathbf{C}_X = E[(\mathbf{X}-\overline{\mathbf{X}})(\mathbf{X}-\overline{\mathbf{X}})^t] = \mathbf{R}_X - (\overline{\mathbf{X}})(\overline{\mathbf{X}})^t$$

For a wide sense stationary process,

$$\left.\begin{aligned} \sigma_i^2 = \sigma_j^2 = \sigma^2 \\ \rho_{ij} = \rho_{|i-j|} \end{aligned}\right\} \text{ for } i,j = 1,2,\ldots,N \qquad \mathbf{C}_X = \sigma^2 \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{N-1} \\ \rho_1 & 1 & \cdots & \rho_{N-2} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{N-1} & \rho_{N-2} & \cdots & 1 \end{bmatrix}$$

A Toeplitz matrix

Revisit to W-S Gaussian Processes

$$\mathbf{X} = [X(t_1) \quad X(t_2) \quad \cdots \quad X(t_N)]^t = [X_1 \quad X_2 \quad \cdots \quad X_N]^t$$

$$f_{\mathbf{X}}(x_1,x_2,\cdots,x_N) = \frac{\left|\mathbf{C}_X^{-1}\right|^{1/2}}{(2\pi)^{N/2}} \exp\left\{-\frac{(\mathbf{X}-\overline{\mathbf{X}})^t \mathbf{C}_X^{-1}(\mathbf{X}-\overline{\mathbf{X}})}{2}\right\}$$

$$\mathbf{X}-\overline{\mathbf{X}} = \begin{bmatrix} x_1-\overline{X}_1 \\ x_2-\overline{X}_2 \\ \vdots \\ x_N-\overline{X}_N \end{bmatrix} = \begin{bmatrix} x_1-\overline{X} \\ x_2-\overline{X} \\ \vdots \\ x_N-\overline{X} \end{bmatrix} \qquad \mathbf{C}_X = \sigma^2 \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{N-1} \\ \rho_1 & 1 & \cdots & \rho_{N-2} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{N-1} & \rho_{N-2} & \cdots & 1 \end{bmatrix}$$

When $N = 2$,

$$\mathbf{C}_X = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad \mathbf{C}_X^{-1} = \frac{1}{(1-\rho^2)\sigma^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \qquad \left|\mathbf{C}_X^{-1}\right| = [\sigma^4(1-\rho^2)]^{-1}$$

$$f_{\mathbf{X}}(x_1,x_2) = \frac{1}{2\pi\sigma^2\sqrt{(1-\rho^2)}} \exp\left\{-\frac{(x_1-\overline{X})^2 - 2\rho(x_1-\overline{X})(x_2-\overline{X}) + (x_2-\overline{X})^2}{2\sigma^2(1-\rho^2)}\right\}$$

Crosscorrelation Matrix

$$\mathbf{X}(t) = [X_1(t) \quad X_2(t) \quad \cdots \quad X_N(t)]^t$$

$$\mathbf{Y}(t) = [Y_1(t) \quad Y_2(t) \quad \cdots \quad Y_N(t)]^t$$

$\mathbf{X}$ and $\mathbf{Y}$ can be considered as signals transmitted and received (at time t) by multiple antennas at N different points.

We can define a crosscorrelation matrix between $\mathbf{X}$ and $\mathbf{Y}$,

$$\mathbf{R_{XY}}(\tau) = E[\mathbf{X}(t)\mathbf{Y}^t(t+\tau)] = \begin{bmatrix} R_{11}(\tau) & R_{12}(\tau) & \cdots & R_{1N}(\tau) \\ R_{21}(\tau) & R_{22}(\tau) & \cdots & R_{2N}(\tau) \\ \cdots & \cdots & \cdots & \cdots \\ R_{N1}(\tau) & R_{N2}(\tau) & \cdots & R_{NN}(\tau) \end{bmatrix}$$

Sometimes, $\mathbf{X}$ and $\mathbf{Y}$ may not have the same number of dimensions. The crosscorrelation matrix in that case then is not a square matrix.