ECE 3075A
Random Signals

Lecture 29
Spectral Density

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Spectrum

- Frequency-domain representation of a time function.
- Based on the principle of decomposing a time function into sum of (complex) sinusoids.
- The mathematical tool for doing such decomposition is the Fourier transform:

$$F_X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_X(\omega) e^{j\omega t} d\omega$$

Fourier transform Inverse Fourier transform

$F_X(\omega)$ (a complex quantity) is the relative magnitude and phase of a steady-state (complex) sinusoid of frequency ω. When $F_X(\omega)$ at all the frequencies are summed together, we have the original time function or signal, as seen in the inverse transform.

Amplitude Spectrum and Phase Spectrum

$$F_X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad F_X(\omega) = |F_X(\omega)| e^{j\angle F_X(\omega)}$$

$$\ln F_X(\omega) = \ln |F_X(\omega)| + j\angle F_X(\omega)$$

$\mathrm{Re}\{\ln F_X(\omega)\} = \ln |F_X(\omega)|$, log-magnitude spectrum

$\mathrm{Im}\{\ln F_X(\omega)\} = \angle F_X(\omega)$, phase spectrum

$|F_X(\omega)|$ is the amplitude of the sinusoid of frequency ω, and as a function of ω, is called the **amplitude or magnitude density** of the signal $x(t)$. It gives an indication how the energy of $x(t)$ is distributed across the frequency range, possibly from $-\infty$ to ∞.

When the function is a random process, in order for $|F_X(\omega)|$ to exist, or $X(t)$ to be Fourier transformable, $X(t)$ has to be absolute-summable: $\int_{-\infty}^{\infty} |X(t)| dt < \infty$

A wide sense stationary process with non-zero mean is not absolutely summable. Why?

Fourier Transform of A Random Process

- Sample functions of a wide-sense stationary process are usually not absolutely summable and therefore are not Fourier transformable.

  Absolute summability requires: $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

- Need to modify the process so that the transform exists; use a truncated process:

$$X_T(t) = \begin{cases} X(t), & |t| \le T < \infty \\ 0, & |t| > T \end{cases} \Rightarrow \quad \int_{-\infty}^{\infty} |X_T(t)| dt < \infty, \quad \int_{-\infty}^{\infty} |X_T(t)|^2 dt < \infty$$

Then its Fourier transform is

$$F_{X_T}(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t} dt$$

This is an intermediate step for us to derive the power density spectrum of a random process.

Power Density Spectrum

$$F_{X_T}(\omega)=\int_{-\infty}^{\infty}X_T(t)e^{-j\omega t}dt=\int_{-T}^{T}X(t)e^{-j\omega t}dt$$

From Parseval's theorem, (signal energy calculated in the time domain is the same as that calculated in the frequency domain)

$$\int_{-T}^{T}X_T^2(t)dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F_{X_T}(\omega)|^2\,d\omega$$

In terms of power: $\frac{1}{2T}\int_{-T}^{T}X_T^2(t)dt=\frac{1}{4\pi T}\int_{-\infty}^{\infty}|F_{X_T}(\omega)|^2\,d\omega$

Taking expectation: $E\left[\frac{1}{2T}\int_{-T}^{T}X_T^2(t)dt\right]=E\left\{\frac{1}{4\pi T}\int_{-\infty}^{\infty}|F_{X_T}(\omega)|^2\,d\omega\right\}$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[X_T^2(t)]dt=\lim_{T\to\infty}\frac{1}{4\pi T}\int_{-\infty}^{\infty}E[|F_{X_T}(\omega)|^2]d\omega$$

Since $E[X_T^2(t)]=\overline{X^2}$ for $-T\le t\le T$, $\overline{X^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T}d\omega$

Define spectral density: $S_X(\omega)=\lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T}$

Spectral Density

$$S_X(\omega)=\lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T}\qquad \overline{X^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)d\omega$$

Since $S_X(\omega)$ is an average over time, it is thus usually called a **power density spectrum**. When $S_X(\omega)$ is integrated over the entire frequency range, we obtain the average power of the signal, which is equal to the mean-square value of the wide-sense stationary process.

Example: $S_X(\omega)=\frac{2a}{\omega^2+a^2}$ $\qquad S_X(0)=\frac{2a}{0^2+a^2}=\frac{2}{a}$

$$\overline{X^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{\omega^2+a^2}d\omega=\frac{1}{2\pi}\left[\frac{2a}{a}\tan^{-1}\left(\frac{\omega}{a}\right)\right]_{-\infty}^{\infty}=\frac{2}{2\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=1$$

Useful integral:

$$\frac{1}{2\pi}\int_p^q\frac{2a}{\omega^2+a^2}d\omega=\frac{1}{\pi}\left[\tan^{-1}\left(\frac{\omega}{a}\right)\right]_p^q\qquad \int_p^q\frac{2a}{(2\pi f)^2+a^2}df=\frac{1}{\pi}\left[\tan^{-1}\left(\frac{2\pi f}{a}\right)\right]_p^q$$

ω in radian/s $\qquad$ f in cycle/s or Hz

Example 7-2.2

A stationary random process has a two-sided spectral density given by

$$S_X(\omega)=\frac{24}{\omega^2+16}\quad \mathrm{V^2/Hz}$$

a. Find the mean-square value of the process;
b. Find the power of the process in the frequency range of ±1 Hz centered at the origin.

$$\overline{X^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{3\times 8}{\omega^2+4^2}d\omega=\frac{3}{\pi}\left[\tan^{-1}\left(\frac{\omega}{4}\right)\right]_{-\infty}^{\infty}=\frac{3}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=3\ \ (\mathrm{V}^2)$$

$$\overline{X_1^2}=\int_{-1}^{1}S_X(f)df=\int_{-1}^{1}\frac{3\times 8}{(2\pi f)^2+4^2}df=\frac{3}{\pi}\left[\tan^{-1}\left(\frac{2\pi f}{4}\right)\right]_{-1}^{1}$$

$$=\frac{3}{\pi}\left[\tan^{-1}\left(\frac{\pi}{2}\right)+\tan^{-1}\left(\frac{\pi}{2}\right)\right]=1.91728\ \ (\mathrm{V}^2)$$

Properties of Spectral Density

- Spectral density is a real, non-negative and even function of frequency (ω or $f[req]$).
- Since it is an even function of the frequency, a rational spectral density of the form

$$S_X(\omega)=\frac{S_0(\omega^{2n}+a_{2n-2}\omega^{2n-2}+\cdots+a_2\omega^2+a_0)}{\omega^{2m}+b_{2m-2}\omega^{2m-2}+\cdots+b_2\omega^2+b_0}$$

  contains only even powers of ω.
- White noise is a process whose spectral density is a constant over the entire frequency range, i.e.,

  $S_X(\omega)=S_0$ for all ω

  (just like a "white" light that contains all colored lights with equal intensity). When the marginal distribution is Gaussian, we call it a **Gaussian white noise**.

Spectral Density of Constant or Periodic Signals

Consider a process $X(t) = A + B\cos(2\pi f_0 t + \Theta)$

where A, B and f_0 are constant and Θ is a random variable uniformly distributed over $(0,2\pi)$.

Let $X_T(t)$ be a truncated version of $X(t)$ over $(-T,T)$.

$$F_{X_T}(f) = \int_{-T}^{T}[A + B\cos(2\pi f_0 + \Theta)]e^{-j2\pi ft}dt$$

$$\boxed{S_X(\omega) = \lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T}}$$

$$\mathbf{F}[X(t)] = \int_{-\infty}^{\infty}[A + B\cos(2\pi f_0 + \Theta)]e^{-j2\pi ft}dt$$

$$= A\delta(f) + (B/2)\left[\delta(f+f_0)e^{-j\Theta} + \delta(f-f_0)e^{j\Theta}\right]$$

$$F_{X_T}(f) = 2T\mathrm{sinc}(2Tf) * \mathbf{F}[X(t)]$$

$$= 2AT\mathrm{sinc}(2Tf) + BT\{\mathrm{sinc}[2(f+f_0)T]e^{-j\Theta} + \mathrm{sinc}[2(f-f_0)T]e^{j\Theta}\}$$

$$|F_{X_T}(f)|^2 = F_{X_T}(f)F^*_{X_T}(f)$$

$$= (2AT)^2\mathrm{sinc}^2(2Tf) + (BT)^2\left\{\mathrm{sinc}^2[2(f+f_0)T] + \mathrm{sinc}^2[2(f-f_0)T]\right\}$$

$$+ C(f)e^{-j\Theta} + C(-f)e^{j\Theta} + D(f)e^{-j2\Theta} + D(-f)e^{j2\Theta}$$

Spectral Density of Constant or Periodic Signals

$$E\left[|F_{X_T}(f)|^2\right]$$

$$= (2AT)^2\mathrm{sinc}^2(2Tf) + (BT)^2\left\{\mathrm{sinc}^2[2(f+f_0)T] + \mathrm{sinc}^2[2(f-f_0)T]\right\}$$

because $E_\Theta[G(f)e^{-j\Theta}] = 0$ and $E_\Theta[H(f)e^{-j2\Theta}] = 0$ for Θ uniformly distributed in $(0,2\pi)$

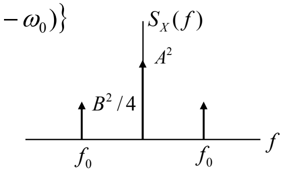
$$S_X(f) = \lim_{T\to\infty}\frac{E\left[|F_{X_T}(f)|^2\right]}{2T}$$

$$= \lim_{T\to\infty}\left\{A^2 2T\mathrm{sinc}^2(2Tf) + (B^2/4)\left(2T\mathrm{sinc}^2[2(f+f_0)T] + 2T\mathrm{sinc}^2[2(f-f_0)T]\right)\right\}$$

But, $\lim_{T\to\infty} 2T\mathrm{sinc}^2(2Tf) = \delta(f)$

Therefore, $S_X(f) = A^2\delta(f) + (B^2/4)\{\delta(f+f_0) + \delta(f-f_0)\}$

And, $S_X(\omega) = 2\pi A^2\delta(\omega) + (\pi B^2/2)\{\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\}$



The spectral density thus consists of three spikes (delta functions) at DC (with height A^2) and at $\pm f_0$ (with height $B^2/4$), respectively.