ECE 3075A

Random Signals

Lecture 31

Spectral Density and Autocorrelation Functions

School of Electrical and Computer Engineering
Georgia Institute of Technology
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Spectral Density & Autocorrelation Function

$$F_{X_T}(\omega)=\int_{-\infty}^{\infty}X_T(t)e^{-j\omega t}dt \qquad F_{X_T}^*(\omega)=\int_{-\infty}^{\infty}X_T(t)e^{j\omega t}dt=F_{X_T}(-\omega)$$

$$S_X(\omega)=\lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T}=\lim_{T\to\infty}\frac{1}{2T}E\left\{\int_{-\infty}^{\infty}X_T(t_1)e^{+j\omega t_1}dt_1\int_{-\infty}^{\infty}X_T(t_2)e^{-j\omega t_2}dt_2\right\}$$

$$S_X(\omega)=\lim_{T\to\infty}\frac{1}{2T}E\left\{\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}X_T(t_1)X_T(t_2)e^{+j\omega(t_2-t_1)}dt_2\right\}$$

$$=\lim_{T\to\infty}\frac{1}{2T}\left\{\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}E[X_T(t_1)X_T(t_2)]e^{+j\omega(t_2-t_1)}dt_2\right\}$$

$$E[X_T(t_1)X_T(t_2)]=\begin{cases}R_X(t_1,t_2), & |t_1|,|t_2|\le T\\ 0, & \text{elsewhere}\end{cases}$$

$$S_X(\omega)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T-t_1}^{T-t_1}d\tau\int_{-T}^{T}R_X(t_1,t_1+\tau)e^{-j\omega\tau}dt_1$$

$$=\int_{-\infty}^{\infty}\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_X(t_1,t_1+\tau)dt_1\right\}e^{-j\omega\tau}d\tau=\int_{-\infty}^{\infty}\langle R_X(t_1,t_1+\tau)\rangle e^{-j\omega\tau}d\tau$$

Spectral Density of A Stationary Process

If $X(t)$ is a stationary ergodic process $\langle R_X(t_1,t_1+\tau)\rangle=R_X(\tau)$

$$S_X(\omega)=\int_{-\infty}^{\infty}R_X(\tau)e^{-j\omega\tau}d\tau$$

The spectral density is the Fourier transform of the autocorrelation function. The autocorrelation function is the inverse Fourier transform of the spectral density function.

$$R_X(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)e^{j\omega\tau}d\omega$$

Since the autocorrelation function is a real, even function,

$$S_X(\omega)=\int_{-\infty}^{\infty}R_X(\tau)(\cos\omega\tau-j\sin\omega\tau)d\tau=2\int_0^{\infty}R_X(\tau)\cos\omega\tau d\tau$$

$$R_X(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)(\cos\omega\tau+j\sin\omega\tau)d\omega=\frac{1}{\pi}\int_0^{\infty}S_X(\omega)\cos\omega\tau d\omega$$

Example – Exponential Autocorrelation

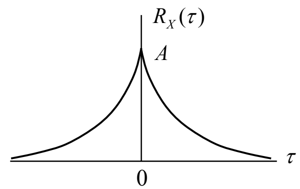
$$R_X(\tau)=Ae^{-\beta|\tau|} \qquad A>0,\ \beta>0$$

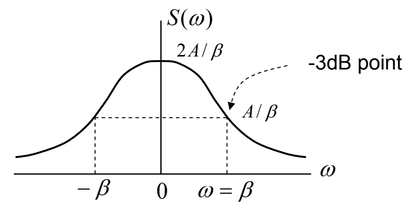
$$S_X(\omega)=\int_{-\infty}^{0}Ae^{\beta\tau}e^{-j\omega\tau}d\tau+\int_0^{\infty}Ae^{-\beta\tau}e^{-j\omega\tau}d\tau=A\frac{e^{(\beta-j\omega)\tau}}{\beta-j\omega}\Bigg|_{-\infty}^{0}+A\frac{e^{-(\beta+j\omega)\tau}}{-(\beta+j\omega)}\Bigg|_{0}^{\infty}$$

$$=A\left[\frac{1}{\beta-j\omega}+\frac{1}{\beta+j\omega}\right]=\frac{2\beta A}{\beta^2+\omega^2}$$

$$S_X(\omega=0)=\frac{2\beta A}{\beta^2+0^2}=\frac{2A}{\beta} \qquad S_X(\omega=\beta)=\frac{2\beta A}{\beta^2+\beta^2}=\frac{A}{\beta}=\frac{1}{2}S_X(0)$$

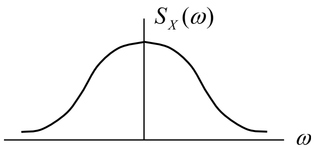
$$\frac{S_X(\beta)}{S_X(0)}=\frac{1}{2}$$ Or in dB: $$10\log_{10}\frac{S_X(\beta)}{S_X(0)}=10\log_{10}\frac{1}{2}=-3.0103\text{ dB}$$





Bandwidth of Spectral Density

When $X(t)$ has a spectral density with components clustered around $\omega = 0$, it is called a lowpass process (signal) or a baseband process (signal).



Given $S_X(\omega)$, which is non-negative and real, one can (heuristically) treat

$$p(\omega) = \frac{S_X(\omega)}{\int_{-\infty}^{\infty} S_X(\omega)d\omega} \qquad \text{note}: \int_{-\infty}^{\infty} p(\omega)d\omega = 1$$

as a function similar to a probability density function.

Then, we can measure the spread of the power density by calculating its second moment (mean-square value):

$$B_{rms}^2 = \int_{-\infty}^{\infty} \omega^2 p(\omega)d\omega = \frac{\int_{-\infty}^{\infty} \omega^2 S_X(\omega)d\omega}{\int_{-\infty}^{\infty} S_X(\omega)d\omega}$$

B_{rms} is called the rms bandwidth (equivalent to std, around mean 0)

Example - Bandwidth

Given the spectral density $S_X(\omega) = \frac{1}{[1+(\omega/10)^2]^2}$

- Find the rms bandwidth;
- Find the frequency at which the spectral density is maximum;
- Find the frequency at which the spectral density is -6 dB from its maximum.

$$\int_{-\infty}^{\infty} S_X(\omega)d\omega = \int_{-\infty}^{\infty} \frac{1}{[1+(\omega/10)^2]^2}d\omega = \int_{-\infty}^{\infty} \frac{10^4}{[10^2+\omega^2]^2}d\omega = 10^4\left\{\frac{\omega}{2\times 10^2(10^2+\omega^2)} + \frac{1}{2\times 10^3}\tan^{-1}\left(\frac{\omega}{10}\right)\right\}\Bigg|_{-\infty}^{\infty} = 5\pi$$

$$B_{rms}^2 = \frac{1}{5\pi}\int_{-\infty}^{\infty} \frac{\omega^2}{[1+(\omega/10)^2]^2}d\omega = \frac{500\pi}{5\pi} = 100, \quad \text{therefore } B_{rms} = 10$$

$S_X(\omega)$ as given is a monotonically decreasing funtion of ω.

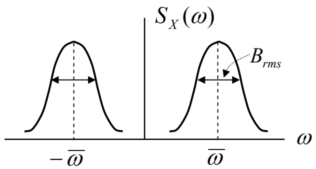
Its maximum occurs at $\omega = 0$ and $S_X(\omega = 0) = 1$.

$$-6 = 10\log_{10} S_X(\omega) - 10\log_{10} S_X(0) = 10\log_{10} S_X(\omega)$$

$$S_X(\omega) = \frac{1}{10^{0.6}} = \frac{1}{4} = \frac{1}{[1+(\omega/10)^2]^2}, \Rightarrow 1+(\omega/10)^2 = 2 \Rightarrow \omega/10 = 1 \Rightarrow \omega = 10 \text{ (rad/s)}$$

Bandwidth of Spectral Density

When $X(t)$ has a spectral density with components clustered around $\omega = \pm\overline{\omega}$, it is called a bandpass process (signal).



The center of the spectral density can be defined as the mean frequency:

$$\overline{\omega} = \int_{-\infty}^{\infty} \omega p(\omega)d\omega = \frac{\int_0^{\infty} \omega S_X(\omega)d\omega}{\int_0^{\infty} S_X(\omega)d\omega}$$

Correspondingly, the rms bandwidth is now

$$B_{rms} = 2\left\{\int_0^{\infty} (\omega-\overline{\omega})^2 p_+(\omega)d\omega\right\}^{1/2} = 2\left\{\frac{\int_0^{\infty} (\omega-\overline{\omega})^2 S_X(\omega)d\omega}{\int_0^{\infty} S_X(\omega)d\omega}\right\}^{1/2}$$

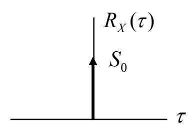
where $p_+(\omega)$ is the positive frequency side of $S_X(\omega)$.

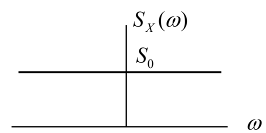
More on White Noise

White noise is a process whose spectral density is a constant for all frequencies: $S_X(\omega) = S_0$

The corresponding autocorrelation function is a delta-function at 0 lag (τ = 0): $R_X(\tau) = S_0\delta(\tau)$

Note: $S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} S_0\delta(\tau)e^{-j\omega\tau}d\tau = S_0$





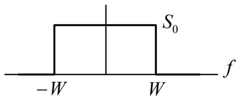
Particular care for the concept of white noise:

In theory: $\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0 d\omega = \infty$

Not a realizable process!!

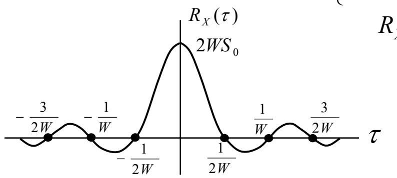
Autocorrelation of Bandlimited White Noise

A more useful concept is the bandlimited white noise whose spectral density is a constant over a finite bandwidth and zero outside the frequency range. For example:



$$S_X(\omega) = \begin{cases} S_0, & |\omega| \le 2\pi W \\ 0, & |\omega| > 2\pi W \end{cases}$$

$$R_X(\tau) = \mathbf{F}^{-1}\{S_X(f)\} = \mathbf{F}^{-1}\left\{S_0 \text{rect}\left(\frac{f}{2W}\right)\right\} = 2WS_0 \text{sinc}(2W\tau)$$



$R_X(\tau) = 0$ at $\tau = n/2W,\ n = \pm 1, \pm 2, \cdots$

Random variables from a band-limited white noise are uncorrelated if they are separated in time by any multiple of $1/2W$ seconds.

Therefore, if a continuous time bandlimited white noise process is sampled at twice the maximum frequency limit ($2W$), then the resultant samples of the discrete time sequence are uncorrelated.