ECE 3075A
Random Signals

Lecture 32

Correlation Functions & Power Density Spectrum, Cross-spectral Density

School of Electrical and Computer Engineering
Georgia Institute of Technology
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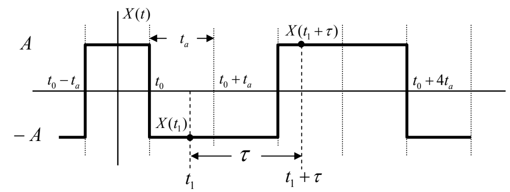
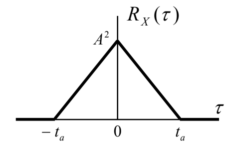
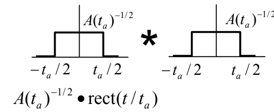
Power Density Spectrum - Applications

- Consider a Binary Random Process; use previously derived autocorrelation function







$$R_X(\tau)=\begin{cases}A^2\left[\dfrac{t_a-|\tau|}{t_a}\right]=A^2\left[1-\dfrac{|\tau|}{t_a}\right], & |\tau|\le t_a\\ 0, & |\tau|>t_a\end{cases}$$

$$S_X(\omega)=2\int_0^{t_a}A^2\left(1-\frac{\tau}{t_a}\right)e^{-j\omega\tau}d\tau$$

$$A(t_a)^{-1/2}\bullet\text{rect}(t/t_a) \overset{\mathbf{F}}{\Leftrightarrow} A(t_a)^{1/2}\text{sinc}(t_a\omega/2\pi)$$

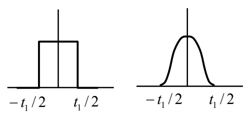
$$=\mathbf{F}[A\bullet\text{Rect}(t/t_a)]\bullet\mathbf{F}[A\bullet\text{Rect}(t/t_a)]=A^2t_a\text{sinc}^2(t_a\omega/2\pi)$$

Alternative View - Spectral Density of Pulses

Consider the following elementary pulses that are often used:

$p_r(t)=\text{rect}(t/t_1)$ and

$$p_c(t)=\frac{1}{2}\left(1+\cos\frac{2\pi t}{t_1}\right),\quad |t|\le\frac{t_1}{2};\ =0,\quad |t|>\frac{t_1}{2}.$$



A process formed with these elementary pulses can be defined:

$$X(t)=\sum_{i=-\infty}^{\infty}G(i)Ap(t-it_1)$$

where $G(i)$ is i.i.d. with $\Pr\{G(i)=-1\}=\Pr\{G(i)=1\}=0.5$, p is either p_r or p_c, and A is a constant that defines the pulse amplitude.

$X(t)$ is thus sum of many independent, zero-mean processes because $E\{G(i)\}=0$. Using previous results, we have

$$R_{X_T}(t,t+\tau)=A^2\sum_{i=-I}^{I}R_P(t-it_1,t-it_1+\tau),\ |\tau|\le t_1/2,\ \text{where } I=T/t_1,$$

and $R_p(t,t+\tau)$ is the autocorrelation of the elementary pulse.

Spectral Density of Pulses

Or more formally, $X_T(t)=\sum_{i=-I}^{I}G(i)Ap(t-it_1)$

$$E[X_T(t)X_T(t')]=E\left[\sum_{i=-I}^{I}G(i)Ap(t-it_1)\sum_{j=-I}^{I}G(j)Ap(t'-jt_1)\right]$$

$$=\sum_{i=-I}^{I}\sum_{j=-I}^{I}E[G(i)G(j)]Ap(t-it_1)Ap(t'-jt_1)$$

$$=\sum_{i=-I}^{I}\sum_{j=-I}^{I}\delta(i-j)Ap(t-it_1)Ap(t'-jt_1)=\sum_{i=-I}^{I}A^2p(t-it_1)p(t'-it_1)$$

$$E[|F_{X_T}(\omega)|^2]=E\left\{\left[\int_{-\infty}^{\infty}X_T(t)e^{-j\omega t}dt\right]\left[\int_{-\infty}^{\infty}X_T(t')e^{j\omega t'}dt'\right]\right\}$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E[X_T(t)X_T(t')]e^{-j\omega t}e^{j\omega t'}dtdt'=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{i=-I}^{I}A^2p(t-it_1)p(t'-it_1)e^{-j\omega t}e^{j\omega t'}dtdt'$$

$$=\sum_{i=-I}^{I}A^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p(t)p(t')e^{-j\omega(t+it_1)}e^{j\omega(t'+it_1)}dtdt'=\sum_{i=-I}^{I}A^2|F_P(\omega)|^2=2IA^2|F_P(\omega)|^2$$

$$=\frac{2T}{t_1}A^2|F_P(\omega)|^2 \qquad \Rightarrow\quad S_X(\omega)=\lim_{T\to\infty}\frac{E[|F_{X_T}(\omega)|^2]}{2T}=\frac{A^2|F_P(\omega)|^2}{t_1}$$

Spectral Density of Binary Processes

- For $p(t) = p_r(t) = \mathrm{rect}(t/t_1)$,

  $F_P(\omega) = t_1 \mathrm{sinc}(t_1\omega/2\pi)$ and thus $|F_P(\omega)|^2 = t_1^2 \mathrm{sinc}^2(t_1\omega/2\pi)$

  Therefore, $$S_X(\omega) = \frac{A^2|F_P(\omega)|^2}{t_1} = A^2 t_1 \mathrm{sinc}^2(t_1\omega/2\pi)$$

- For $p(t) = p_c(t) = \frac{1}{2}\left(1+\cos\frac{2\pi t}{t_1}\right), \quad |t| \le \frac{t_1}{2}; \ = 0, \quad |t| > \frac{t_1}{2}$,

$$F_P(\omega) = \frac{1}{2}\int_{-t_1/2}^{t_1/2}\left(1+\cos\frac{2\pi t}{t_1}\right)e^{-j\omega t}dt = \frac{t_1}{2}\left[\frac{\sin(\omega t_1/2)}{(\omega t_1/2)}\right]\left[\frac{\pi^2}{\pi^2-(\omega t_1/2)^2}\right]$$

$$S_X(\omega) = \frac{A^2 t_1}{4}\left[\frac{\sin(\omega t_1/2)}{(\omega t_1/2)}\right]^2\left[\frac{\pi^2}{\pi^2-(\omega t_1/2)^2}\right]^2$$

$$S_X(f) = \frac{A^2 t_1}{4}\mathrm{sinc}^2(t_1 f)\left[\frac{1}{1-(t_1 f)^2}\right]^2$$

Note that in both cases, max $S_X(\omega)$ occurs at $\omega = 0$

Example

The spectral density of the pulses thus defines the bandwidth of the binary signal carried by these pulses. This bandwidth is a function of the pulse width t_1.

We often need to "match" the bandwidth of the signal to the bandwidth of the transmission channel so as to reduce undesirable distortion of interference.

For example, we may request a bandwidth which would support transmission of the signal up to the frequency at which the spectral density is no more than 1% of its maximum. What would this bandwidth be?

$$\frac{S_X(f)}{S_X(0)} \le 0.01 \quad \text{for } |f| > f_1$$

For the rectangular pulse,

$$S_X(0) = A^2 t_1 \qquad S_X(f_1) = A^2 t_1 \mathrm{sinc}^2(t_1 f_1) = 0.01 A^2 t_1 \Rightarrow \mathrm{sinc}^2(t_1 f_1) = 0.01$$

$$t_1 f_1 \pi = 8.4226 \quad \Rightarrow \quad f_1 = 2.681/t_1$$

For the raised cosine pulse,

$$S_X(0) = A^2 t_1/4$$

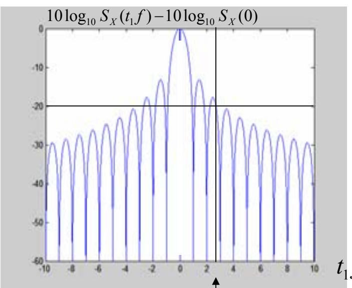
$$S_X(f_1) = \frac{A^2 t_1}{4}\mathrm{sinc}^2(t_1 f_1)\left[\frac{1}{1-(t_1 f_1)^2}\right]^2 = 0.01\frac{A^2 t_1}{4}$$

$$t_1 f_1 \pi = 5.1836 \quad \Rightarrow \quad f_1 = 1.65/t_1$$

Bandwidth of Various Pulses

$$\frac{S_X(f)}{S_X(0)} \le 0.01 \quad \text{for } |f| > f_1 \quad \Rightarrow \quad 10\log_{10}(0.01) = -20\ (\text{dB}) \ \text{at} \ f_1$$

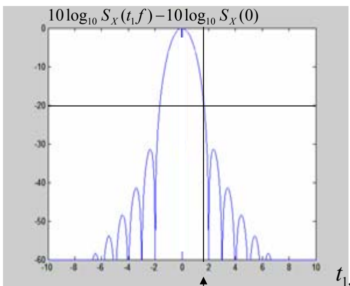
Rectangular pulse



$t_1 f$

2.681

Raised-cosine pulse



$t_1 f$

1.65

Example

An nth- order Butterworth spectrum is one whose spectral density is given by $$S_X(f) = \frac{1}{1+(f/W)^{2n}}$$

in which W is the so-called half-power bandwidth.

1. Find the bandwidth outside of which the spectral density is less than 1% of its maximum value.
2. For n=1, find the bandwidth (F) outside of which no more than 1% of the average power exists.

$$\max S_X = S_X(0) = \frac{1}{1+(0/W)^{2n}} = 1$$

$S_X(f)$ is a monotonically decreasing function of f.

$$S_X(f) = \frac{1}{1+(f/W)^{2n}} = 0.01 \quad \Rightarrow \quad 100 = 1+(f/W)^{2n} \quad \Rightarrow \quad f = W(99)^{1/(2n)}$$

$$\int_{-\infty}^{\infty} S_X(f)df = \int_{-\infty}^{\infty}\frac{1}{1+(f/W)^2}df = W\tan^{-1}\left(\frac{f}{W}\right)\Bigg|_{-\infty}^{\infty} = W\pi$$

$$\int_p^q \frac{2a}{(2\pi f)^2+a^2}df = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{2\pi f}{a}\right)\right]_p^q$$

$$\int_{-F}^{F}\frac{1}{1+(f/W)^2}df = W\tan^{-1}\left(\frac{f}{W}\right)\Bigg|_{-F}^{F} = W2\tan^{-1}\left(\frac{F}{W}\right) = 0.99W\pi \qquad F = 63.657W$$

Cross-Spectral Density

- Just as we are interested in cross-correlation analysis (e.g., to investigate the joint statistical behavior of the input and the output of a system), we are interested in cross-spectral density, which is the frequency domain representation of the cross-correlation function.

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[F_{X_T}(-\omega)F_{Y_T}(\omega)]}{2T} \text{ and } S_{YX}(\omega) = \lim_{T\to\infty} \frac{E[F_{Y_T}(-\omega)F_{X_T}(\omega)]}{2T}$$

X_T and Y_T are truncated processes as defined previously

Key properties:

1. $S_{XY}(\omega) = S^*_{YX}(\omega)$ (* denotes complex conjugate)
2. $\mathrm{Re}\{S_{XY}(\omega)\}$ is an even function of ω. Also true for $S_{YX}(\omega)$.
3. $\mathrm{Im}\{S_{YX}(\omega)\}$ is an odd function of ω. Also true for $S_{YX}(\omega)$.

Cross-spectral Density and Cross-correlation

- Similar to the relationship between autocorrelation function and power spectral density,

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau \qquad S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)e^{-j\omega\tau}d\tau$$

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega \qquad R_{YX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)e^{j\omega\tau}d\omega$$

Example:

For two jointly stationary random processes, the crosscorrelation function is
$$R_{XY}(\tau) = \begin{cases} 2e^{-2\tau}, & \tau > 0 \\ 0, & \tau < 0 \end{cases}$$

The corresponding cross-spectral density is

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau = \int_{0}^{\infty} 2e^{-(j\omega+2)\tau}d\tau = \frac{2\exp[-(j\omega+2)\tau]}{-(j\omega+2)}\Big|_0^\infty = \frac{2}{j\omega+2}$$

$$S_{YX}(\omega) = S^*_{XY}(\omega) = \frac{2}{-j\omega+2}$$