**ECE 3075A**
***Random Signals***

**Lecture 33**

## Spectral Density And Introduction to System Analysis

School of Electrical and Computer Engineering
Georgia Institute of Technology
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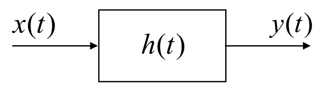
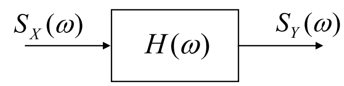
## System Analysis




Time-domain analysis     Frequency-domain analysis

- Only interested in bounded-input/bounded-output systems.
- The output of the system, $y(t)$, is the result of convolution between the input (excitation to the system), $x(t)$ , and the **impulse response**, $h(t)$ , of the system.

$$y(t) = \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)d\lambda$$

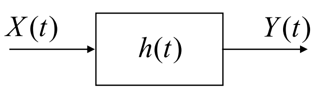
- For the system to be realizable and stable,

$$h(t) = 0 \text{ for } t < 0 \quad \text{(causality) and} \quad \int_{-\infty}^{\infty} |h(t)| dt < \infty \quad \text{(stability)}$$

Thus, $y(t) = \int_0^{\infty} x(t-\lambda)h(\lambda)d\lambda = \int_{-\infty}^{0} x(\lambda)h(t-\lambda)d\lambda$



## Random Input to A System




$x(t) \Rightarrow X(t) \qquad y(t) \Rightarrow Y(t)$

The input is no longer a fixed function.

The same linear system concept applies.

Example: $h(t) = \begin{cases} 5e^{-3t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$X(t) = M + 4\cos(2t + \Theta)$ where M is a random variable and $\Theta$ is an independent random variable, uniformly distributed in (0,2π).

$$Y(t) = \int_{-\infty}^{t} [M + 4\cos(2\lambda + \Theta)] 5e^{-3(t-\lambda)} d\lambda$$

$$= \frac{5}{3} M + \frac{20}{13}[3\cos(2t+\Theta) + 2\sin(2t+\Theta)]$$

$Y(t)$ is also a random process whose statistical properties can be derived from the distributions of the random variables, $M$ and $\Theta$ .



## Example

A linear system has an impulse response of the form

$$h(t) = \begin{cases} te^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and an input that is a random process of the form

$$X(t) = M, \quad -\infty < t < \infty$$

where $M$ is a random variable uniformly distributed in (0,12).

1. Write an expression for the output;
2. Find the mean value of the output;
3. Find the variance of the output.

$$Y(t) = \int_0^{\infty} M\lambda e^{-2\lambda} d\lambda = M \frac{e^{-2\lambda}}{4}(-2\lambda - 1)\Big|_0^{\infty} = \frac{M}{4}$$

$$E[Y(t)] = E[M/4] = 6/4 = 3/2$$

$$E[Y^2(t)] = E[M^2/16] = (144/3)/16 = 3$$

$$\sigma_Y^2 = E[Y^2(t)] - \{E[Y(t)]\}^2 = 3 - (9/4) = 3/4$$

# Example

A linear system has an impulse response of the form

$$h(t)=\begin{cases}5\delta(t)+3, & 0\le t<1\\ 0, & \text{elsewhere}\end{cases}$$

The input process is of the form $X(t)=2\cos(2\pi t+\Theta),\quad -\infty<t<\infty$ where $\Theta$ is a random variable uniformly distributed in $(0,2\pi)$.

1. Write an expression for the output process;
2. Find the mean value of the output;
3. Find the variance of the output.

$$Y(t)=\int_0^1 2\cos[2\pi(t-\lambda)+\Theta][5\delta(\lambda)+3]d\lambda$$

$$=10\cos(2\pi t+\Theta)-\frac{6}{2\pi}\sin[2\pi(t-\lambda)+\Theta]\Big|_{\lambda=0}^{1}=10\cos(2\pi t+\Theta)$$

$$E[Y(t)]=E_\Theta[10\cos(2\pi t+\Theta)]=0$$

$$E[Y^2(t)]=E_\Theta[100\cos^2(2\pi t+\Theta)]=E_\Theta[50+50\cos(4\pi t+2\Theta)]=50$$

$$\sigma_Y^2=E[Y^2(t)]-\{E[Y(t)]\}^2=50-0=50$$

# Mean of System Output

As demonstrated in the previous examples,

$$\overline{Y}=E[Y(t)]=E\left[\int_{-\infty}^{\infty}X(t-\lambda)h(\lambda)d\lambda\right]$$

In general, for $E\left[\int_{t_1}^{t_2}Z(t)f(t)dt\right]=\int_{t_1}^{t_2}E[Z(t)]f(t)dt$ , it requires that

1. $\int_{t_1}^{t_2}E[|Z(t)|]\,|f(t)|\,dt<\infty$
2. $Z(t)$ is bounded on the interval $(t_1,t_2)$ .

In most cases, we assume these conditions are satisfied.

And if $X(t)$ is wide sense stationary with $E[X(t)]=\overline{X}$ , then

$$\overline{Y}=\int_{-\infty}^{\infty}E[X(t-\lambda)]h(\lambda)d\lambda=\overline{X}\int_{-\infty}^{\infty}h(\lambda)d\lambda$$

Note that $\int_{-\infty}^{\infty}h(\lambda)d\lambda$ is the dc gain of the system; the dc component of the output is thus equal to the dc component of the input times the dc gain of the system.

# Mean-Square Value of System Output

$$\overline{Y^2}=E[Y^2(t)]=E\left[\int_0^\infty X(t-\lambda_1)h(\lambda_1)d\lambda_1\bullet\int_0^\infty X(t-\lambda_2)h(\lambda_2)d\lambda_2\right]$$

$$=E\left[\int_0^\infty d\lambda_1\int_0^\infty X(t-\lambda_1)X(t-\lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_2\right]$$

$$=\int_0^\infty d\lambda_1\int_0^\infty E[X(t-\lambda_1)X(t-\lambda_2)]h(\lambda_1)h(\lambda_2)d\lambda_2$$

But, $E[X(t-\lambda_1)X(t-\lambda_2)]=R_X(t-\lambda_1-t+\lambda_2)=R_X(\lambda_2-\lambda_1)$

Therefore, $\overline{Y^2}=\int_0^\infty d\lambda_1\int_0^\infty R_X(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2$

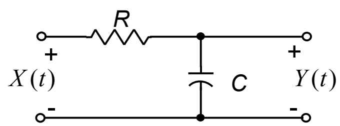
Example: For white noise input with $R_X(\tau)=\frac{N_0}{2}\delta(\tau)$, the output "noise" power is

$$\overline{Y^2}=\int_0^\infty d\lambda_1\int_0^\infty R_X(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2$$

$$=\int_0^\infty d\lambda_1\int_0^\infty \frac{N_0}{2}\delta(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2=\frac{N_0}{2}\int_0^\infty h^2(\lambda)d\lambda$$

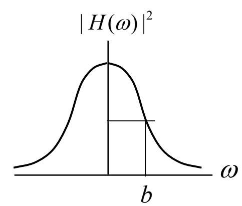
  

# Analysis of A Simple System




$$h(t)=\begin{cases}be^{-bt}, & t\ge 0\\ 0, & t<0\end{cases}$$

where $b=\frac{1}{RC}$




$$\overline{Y}=\overline{X}\int_0^\infty be^{-b\lambda}d\lambda=\overline{X}\frac{be^{-b\lambda}}{-b}\Big|_0^\infty=\overline{X}$$

When the input is white noise with $R_X(\tau)=\frac{N_0}{2}\delta(\tau)$

$$\overline{Y^2}=\frac{N_0}{2}\int_0^\infty b^2e^{-2b\lambda}d\lambda=b^2\frac{N_0}{2}\frac{e^{-2b\lambda}}{-2b}\Big|_0^\infty=\frac{bN_0}{4}$$

$$\mathbf{F}\{h(t)\}=H(\omega)=\frac{b}{b+j\omega}\quad(\text{see Table A5});\quad |H(\omega)|^2=\frac{b}{b+j\omega}\frac{b}{b-j\omega}=\frac{b^2}{b^2+\omega^2}$$

$|H(\omega=b)|^2=\frac{b^2}{2b^2}=\frac{1}{2}=\frac{1}{2}|H(\omega=0)|^2$ ⇨ $b$ is the half-power bandwidth of the system

Let $W_{1/2}=\frac{b}{2\pi}$ Hz, then $\overline{Y^2}=\pi W_{1/2}\frac{N_0}{2}$

# Example

A linear system has an impulse response $h(t) = te^{-2t}u(t)$ where $u(t)$ is the unit step function. The input to this system is a white noise process having a 2-sided spectral density of $2\ \mathrm{V}^2/\mathrm{Hz}$ plus a dc component of 2 V.

1. Find the mean value of the output of the system;
2. Find the variance of the output;
3. Find the mean-square value of the output.

$X(t) = 2 + V(t)$ with $R_V(\tau) = 2\delta(\tau)$, $\overline{X} = 2 + E[V(t)] = 2$ and $R_X(\tau) = 4 + 2\delta(\tau)$

$$\overline{Y} = \overline{X}\int_0^\infty te^{-2t}dt = 2\frac{e^{-2t}}{4}(-2t-1)\Big|_0^\infty = \frac{1}{2}$$

$$\overline{Y^2} = \int_0^\infty d\lambda_1 \int_0^\infty R_X(\lambda_2-\lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 = \int_0^\infty d\lambda_1 \int_0^\infty [4+2\delta(\lambda_2-\lambda_1)]h(\lambda_1)h(\lambda_2)d\lambda_2$$

$$= 4\left[\int_0^\infty te^{-2t}dt\right]^2 + 2\int_0^\infty t^2e^{-4t}dt = 4\left[\frac{e^{-2t}}{4}(-2t-1)\Big|_0^\infty\right]^2 + 2\frac{e^{-4t}}{-64}(16t^2+8t+2)\Big|_0^\infty$$

$$= \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$$

$$\sigma_Y^2 = \overline{Y^2} - \overline{Y}^2 = \frac{5}{16} - \frac{1}{4} = \frac{1}{16}$$