ECE 3075A
Random Signals

Lecture 7
Random Variables, Probability Distribution Functions

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Random Variables

- A random variable is a numeric function of the outcome of a random experiment.

  $$S = \{A\}, \qquad X : A \Rightarrow X(A)$$

  Or, equivalently, we consider a probability space defined over the real line. For example, in 2-coin tossing,

  $$S = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$$

  Define $X(\text{HH}) = 0 \quad X(\text{HT}) = 1 \quad X(\text{TH}) = 3 \quad X(\text{TT}) = 6$

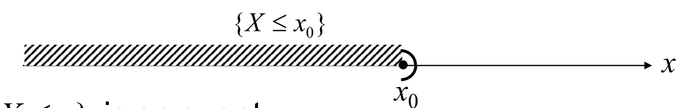
  Then $S_X = \{0, 1, 3, 6\}$

  And $\Pr(0) = \Pr(\text{HH}) = 0.25, \quad \Pr(1) = \Pr(\text{HT}) = 0.25$
  $\Pr(3) = \Pr(\text{TH}) = 0.25, \quad \Pr(6) = \Pr(\text{TT}) = 0.25$

Random Variables (cont'd)

- Types of random variables
  - Discrete random variables: X as a numeric function assumes (equivalently the observation space S_X has) a finite set of values.
  - Continuous random variables: X maps the (infinite number of) events to a continuous range of values.
- Purpose of random variables?
  - Many signals are numeric functions of time; uncertainty in the value of a signal function can be addressed with the theory of probability.
  - To bridge between calculus and set operations (σ algebra); being able to do so broadens our ability to handle the assignment of probability to (infinite) event.

Events Associated with R.V.s



$\{X \le x\}$ is an event.

- X is a random variable, the value of which is to be observed through trials;
- x is an arbitrary real number;
- The event $\{X \le x\}$ means that the observed value of the random variable X is less than or equal to the specified value x;
- A real value x thus defines an event;
- The **probability** of this event is $\Pr(X \le x)$ which is a **function of** x.

Probability Distribution Function

- $\Pr(X \le x)$ is a **function** of x.

  $F_X(x) = \Pr(X \le x)$ is called a **probability distribution function** defined over all x.

  $\{X \le -\infty\} = \phi, \quad F_X(-\infty) = 0$

  $\{X \le \infty\}$ is always true, a sure event, thus, $\quad F_X(\infty) = 1$

  $\{X \le -\infty\} \subset \{X \le x\} \subset \{X \le \infty\}$ for $-\infty < x < \infty$

  $\Rightarrow \quad 0 \le F_X(x) \le 1$

» If $x_1 < x_2, \quad \{X \le x_1\} \subset \{X \le x_2\}, \quad \{X \le x_1\} \cap \{X \le x_2\} = \{X \le x_1\}$

$\{X \le x_2\} = \{X \le x_1\} \cup \{x_1 < X \le x_2\}$, and $\{X \le x_1\} \cap \{x_1 < X \le x_2\} = \phi$

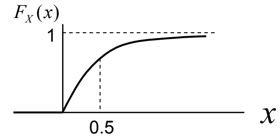
$\therefore \quad \Pr(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

$F_X(x)$ is a non-decreasing function of x.

Example

A random variable has a probability distribution function given by

$$F_X(x) = 0 \qquad -\infty < x \le 0$$
$$= 1 - e^{-2x} \qquad 0 \le x < \infty$$



Find

a) The probability that $X > 0.5$

$$\Pr(X > 0.5) = 1 - F_X(0.5) = 1 - (1 - e^{-1}) = e^{-1} = 0.3679$$

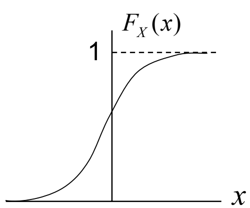
b) The probability that $X \le 0.25$

$$\Pr(X \le 0.25) = F_X(0.25) = 1 - e^{-0.5} = 1 - 0.6065 = 0.3935$$

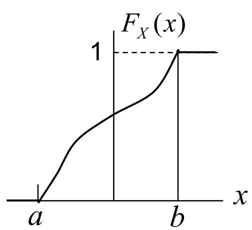
c) The probability that $0.3 < X \le 0.7$

$$\Pr(0.3 < X \le 0.7) = F_X(0.7) - F_X(0.3)$$
$$= 1 - e^{-1.4} - (1 - e^{-0.6}) = 0.5488 - 0.2466 = 0.3022$$

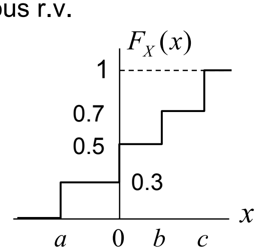
Example of Probability Distribution Function



Continuous r.v.



Continuous r.v. within limits a and b



Discrete r.v. assuming values a, 0, b and c

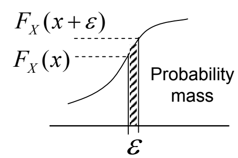
Probability Density Functions

- The slope of the probability distribution function at x represents the incremental probability at that point and thus gives the sense of how likely $X = x$ might be.

$$f_X(x) = \lim_{\varepsilon \to 0} \frac{F_X(x+\varepsilon) - F_X(x+\varepsilon)}{\varepsilon} = \frac{dF_X(x)}{dx}$$

$$f_X(x)dx = \Pr(x < X \le x + dx)$$

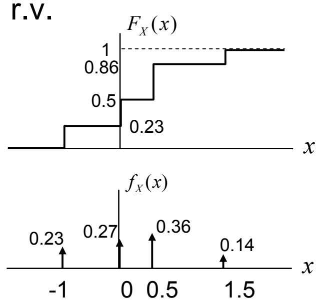
is the **probability mass** at x.



- The derivative is called the probability density function (pdf). Pdf is **non-negative**. In the case of discrete distributions, the pdf consists of Dirac delta functions at those realizable values, each having an area equal to the corresponding magnitude of probability.

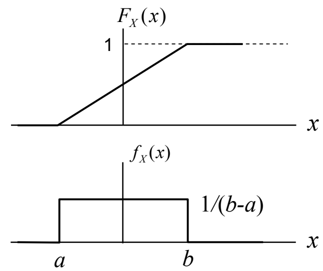
Examples of PDFs

- Distribution of a discrete r.v.



$$f_X(x) = \sum_i p_i \delta(x - x_i)$$

- Continuous r.v. – **uniform distribution**



$$f_X(x) = (b-a)^{-1}, \quad a \le x \le b$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = F_X(\infty) = 1$$

Example

- A random variable X has pdf of the form:

  $f_X(x) = ae^{-bx}u(x)$

  where $u(x)$ is the step function,

$$u(x) = \begin{cases} 1, & \text{for } x \ge 0 \\ 0, & \text{for } x < 0 \end{cases}$$

Find the value of a and b such that this is a valid pdf.

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^{\infty} ae^{-bx}dx = -\frac{ae^{-bx}}{b}\bigg|_0^{\infty} = \frac{a}{b} = 1$$

Therefore, $a = b$