ECE 3075A
Random Signals

Lecture 8

Probability Distribution Functions, Probability Density Functions

School of Electrical and Computer Engineering
Georgia Institute of Technology
Fall, 2003

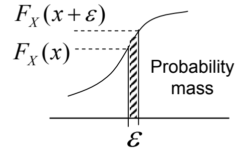
Probability Density Functions

- The slope of the probability distribution function at x represents the incremental probability at that point and thus gives the sense of how likely $X = x$ might be.

$$f_X(x) = \lim_{\varepsilon \to 0} \frac{F_X(x+\varepsilon) - F_X(x+\varepsilon)}{\varepsilon} = \frac{dF_X(x)}{dx}$$

$$f_X(x)dx = \Pr(x < X \le x + dx)$$

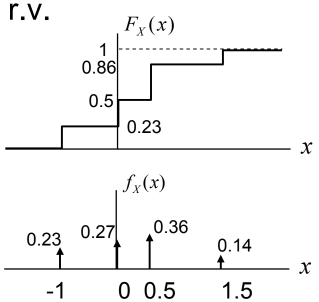
is the **probability mass** at x.



- The derivative is called the probability density function (pdf). Pdf is **non-negative**. In the case of discrete distributions, the pdf consists of Dirac delta functions at those realizable values, each having an area equal to the corresponding magnitude of probability.

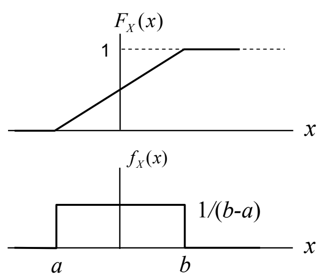
Examples of PDFs

- Distribution of a discrete r.v.



$$f_X(x) = \sum_i p_i \delta(x - x_i)$$

- Continuous r.v. – **uniform distribution**



$$f_X(x) = (b-a)^{-1}, \quad a \le x \le b$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = F_X(\infty) = 1$$

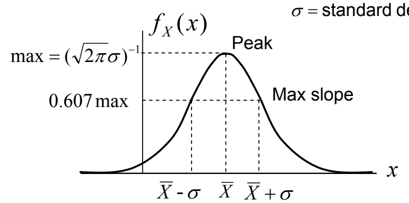
Gaussian Random Variable

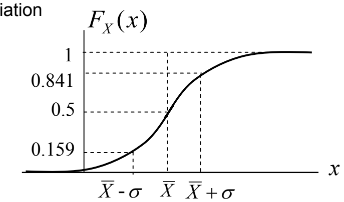
- A r.v. X is gaussian if its pdf is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\overline{X})^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

where $\overline{X}$ and σ^2 are called the mean and variance, respectively.

σ = standard deviation





- ✓ Also called normal distribution, denoted as $\mathcal{N}(x; \overline{X}, \sigma^2)$
- ✓ Pdf has a single peak.
- ✓ $\delta(x-\overline{X}) = \lim_{\sigma \to 0} (\sqrt{2\pi}\sigma)^{-1} \exp\left[-(x-\overline{X})^2/(2\sigma^2)\right]$, a good representation for a delta function because a Gaussian pdf is infinitely differentiable.

Gaussian Integral

Show that $\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\overline{X})^2}{2\sigma^2}\right] dx = 1$

$$G = \int_{-\infty}^{\infty} e^{-x^2} dx \qquad G^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$$

Change to polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$

$dxdy = rdrd\theta$ and $x^2 + y^2 = r^2$

$$G^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} rdr = 2\pi \int_0^{\infty} e^{-r^2} dr$$

Further use change of variable : $t = r^2$ and $dt = 2rdr$

$$G^2 = 2\pi \int_0^{\infty} e^{-r^2} rdr = \pi \left[\int_0^{\infty} e^{-t} dt\right] = \pi \qquad G = \sqrt{\pi}$$

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ or equivalently, $\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1$

Gaussian Distribution

- Often expressed in zero mean and unity variance form with

  $u = \frac{x-\overline{X}}{\sigma}, \qquad f_U(u) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-u^2}{2}\right], \quad -\infty < u < \infty$

  Or equivalently, define $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[\frac{-u^2}{2}\right] du,$

  Then, $F_X(x) = \Phi\left(\frac{x-\overline{X}}{\sigma}\right)$

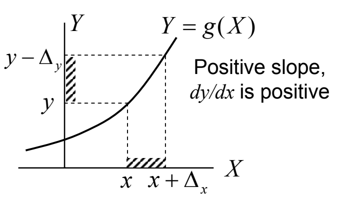
- The Q-function $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left[\frac{-u^2}{2}\right] du = 1 - \Phi(x)$
- The error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left[-u^2\right] du$

$$Q(x) = \frac{1}{2}\left[1 - erf\left(\frac{x}{\sqrt{2}}\right)\right] \approx \left[\frac{1}{(1-a)x + a\sqrt{x^2+b}}\right] \frac{\exp(-x^2/2)}{\sqrt{2\pi}}, \quad x \geq 0$$

$$a = 0.339, \quad b = 5.510$$

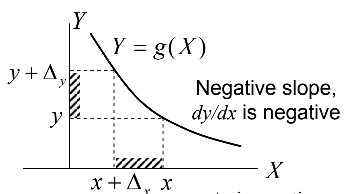
Functions of Random Variable

- X is a random variable with pdf $f_X(x)$.
- Y is a monotonic function of X; $Y = g(X)$. Find $f_Y(y)$.



$\Pr(x < X \leq x + \Delta_x) = \Pr(y < Y \leq y + \Delta_y)$

$f_X(x)dx = f_Y(y)dy$ or $f_Y(y) = f_X(x)\frac{dx}{dy}$



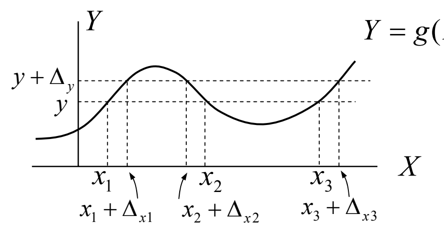
$\Pr(x + \Delta_x < X \leq x) = \Pr(y < Y \leq y + \Delta_y)$

$f_X(x)dx = f_Y(y)(-dy)$ or $f_Y(y) = f_X(x)\frac{-dx}{dy}$

Hence, $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

Expressed in y with $g^{-1}(y) = x, \quad f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$

Non-Monotonic Functions of R.V.



$Y = g(X)$

$y = g(x_1) = g(x_2) = g(x_3)$

$\Pr(y < Y \leq y + \Delta_y)$

$= \Pr(x_1 < X \leq x_1 + \Delta_{x1})$

$+ \Pr(x_2 + \Delta_{x2} < X \leq x_2)$

$+ \Pr(x_3 < X \leq x_3 + \Delta_{x3})$

$$f_Y(y) = \sum_{\text{for all } x = g^{-1}(y)} f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$$

Example:

$Y = X^2$ or $X = \sqrt{Y}, \quad \left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}},$ but for any $y > 0, \; x = \pm\sqrt{y}$

Therefore,

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right], \quad y \geq 0; \qquad = 0, \quad y < 0$$

