ECE 3075A
Random Signals

Lecture 9
Probability Distribution Functions

School of Electrical and Computer Engineering
Georgia Institute of Technology
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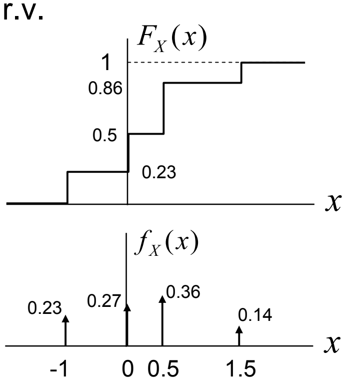
Common Probability Distribution Functions

- Uniform distribution
- Gaussian or normal distribution
- Binomial distribution
- Poisson distribution
- Exponential and double exponential distribution
- Cauchy distribution
- Log normal distribution
- Mixed distribution and mixture density functions

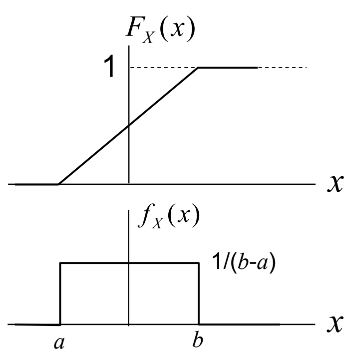
Examples of PDFs

- Distribution of a discrete r.v.



$$f_X(x) = \sum_i p_i \delta(x - x_i)$$

- Continuous r.v. – uniform distribution



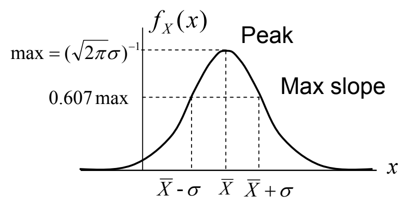
$$f_X(x) = (b-a)^{-1}, \quad a \le x \le b$$

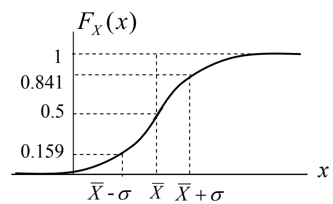
Gaussian Random Variable

- A r.v. X is gaussian if its pdf is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\overline{X})^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

where $\overline{X}$ and σ^2 are called the mean and variance, respectively.





- ✓ Also called normal distribution, denoted as $\mathcal{N}(x; \overline{X}, \sigma^2)$
- ✓ Pdf has single peak.
- ✓ $\delta(x-\overline{X}) = \lim_{\sigma \to 0} (\sqrt{2\pi}\sigma)^{-1} \exp\left[-(x-\overline{X})^2/(2\sigma^2)\right]$, a good representation for a delta function because a gaussian pdf is infinitely differentiable.

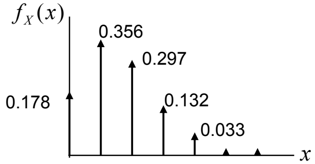
Binomial Distribution

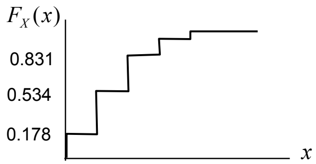
- A discrete distribution

  Binomial density function: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ = binomial coeff.

$$f_X(x) = \sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(x-k) \quad \text{for } 0 < p < 1 \text{ and integer } n.$$

$n = 6,$
$p = 0.25$





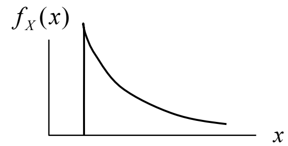
The binomial density function approaches gaussian with mean np and variance $np(1-p)$ when n is large and n, p satisfy certain conditions.

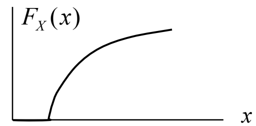
Exponential Density and Distribution

- Exponential pdf and distribution function – for $b>0$ and $-\infty<a<\infty$,

$$f_X(x) = \begin{cases} b^{-1}\exp[-(x-a)/b], & x \ge a \\ 0, & x < a \end{cases}$$

$$F_X(x) = \begin{cases} 1-\exp[-(x-a)/b], & x \ge a \\ 0, & x < a \end{cases}$$





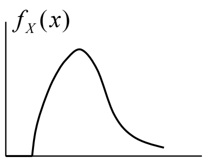
Useful in
- describing raindrop size when a large number of rainstorm measurements are made
- modeling the fluctuation in strength in received radar signal from a certain types of aircraft
- modeling the interval of arrival time of a certain traffic

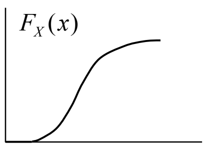
Rayleigh Density and Distribution

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)\exp\left[\frac{-(x-a)^2}{b}\right], & x \ge a \\ 0, & x < a \end{cases}$$

$$F_X(x) = \begin{cases} 1-\exp\left[\frac{-(x-a)^2}{b}\right], & x \ge a \\ 0, & x < a \end{cases}$$

$b > 0, \quad -\infty < a < \infty$





- Was derived as the density function of the envelop of the sum of many sine waves of different frequencies
- Also used in modeling coordinate errors $\Delta_r = \sqrt{\Delta_x^2 + \Delta_y^2}$ where Δ_x and Δ_y are independent gaussian r.v. with zero mean and equal variance.

Poisson Distribution

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \delta(x-k), \quad \lambda > 0$$

Used in describing:
- Number of defective units in a sample taken from a production line over a period of time t
- Number of telephone calls made during a period of time t
- Number of customers coming to a store within a period of time t
- Number of electrons emitted from a section of cathode in a given time interval t

λ is the average number of “arrivals”, “calls”, or “emissions” within a unit time interval.

Other Density & Distribution Functions

- Cauchy $$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

$$F_X(x) = \int_{-\infty}^{x} \frac{dx}{\pi(1+x^2)} = \frac{1}{\pi}\left[\arctan x + \frac{\pi}{2}\right], \quad -\infty < x < \infty$$

- Mixed Distribution or Mixture

$$f_X(x) = \sum_{k=1}^{n} c_k p_k(x), \quad \sum_{k=1}^{n} c_k = 1, \quad \text{and} \quad F_X(x) = \sum_{k=1}^{n} c_k P_k(x)$$

Where $p_k(x)$ is some kernel density function such as Gaussian,

then, $f_X(x) = \sum_{k=1}^{n} c_k \mathcal{N}_k(x; \eta_k, \sigma_k^2)$ and $\overline{X} = \sum_{k=1}^{n} c_k \eta_k$

Functions of Random Variable (Reminder)

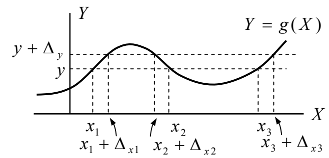
- X is a random variable with pdf $f_X(x)$.
- Y is a monotonic function of X; $Y = g(X)$. Find $f_Y(y)$.

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

Expressed in y with $g^{-1}(y) = x$, $\quad f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$

- If Y is non-monotonic function of X,

$$f_Y(y) = \sum_{\text{for all } x = g^{-1}(y)} f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$$



Example: $Y = X^2$ or $X = \sqrt{Y}$, $\left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}}$, but for any $y > 0$, $x = \pm\sqrt{y}$

Therefore, $f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right]$, $\quad y \ge 0; \qquad = 0, \quad y < 0$