ECE-3075A Random Signals
Summer 2003 - Final Exam

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Problem #1 (7pts ea., 21pts total)

A random variable X has a probability density function of the form $f_X(x) = \exp\{-a|x|\} \quad -\infty < x < \infty$

A second random variable Y is related to X by $Y = |X|$.

a) Find the numeric value of a;

b) Find the probability density function of Y;

c) A third random variable Z is a function of X and Y, i.e., $Z = \frac{X}{Y}$; find the first and second moment of Z.

a) $f_X(x) = \exp\{-a|x|\} \quad -\infty < x < \infty$

$$\int_{-\infty}^{\infty} \exp\{-a|x|\}dx = 2\int_0^{\infty} \exp\{-ax\}dx = \frac{2}{a} = 1 \qquad \text{there fore, } a = 2$$

b) $Y = |X| \qquad \frac{dx}{dy} = \begin{cases} 1, & x \ge 0 \\ -1, & x < 0 \end{cases} \qquad f_X(x) = \exp\{-a|x|\} \quad -\infty < x < \infty$

$$f_Y(y) = (|1|)\exp\{-ay\} + (|-1|)\exp\{-ay\} = 2\exp\{-2y\} \quad 0 \le y < \infty$$

c) $Z = \frac{X}{Y} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

Since $f_X(x) = \exp\{-a|x|\}, -\infty < x < \infty$, is symmetric with respect to 0, $\Pr\{Z = 1\} = \Pr\{Z = -1\} = 0.5$

$$\overline{Z} = E\{Z\} = -1 \times 0.5 + 1 \times 0.5 = 0$$

$$\overline{Z^2} = E\{Z^2\} = 1^2 \times 0.5 + 1^2 \times 0.5 = 1$$

Problem #2 (7pts ea., 21pts total)

The characteristic function of two independent Gaussian random variable, X and Y, are given by $\Phi_X(u) = e^{ju3-(2u^2)}$ and $\Phi_Y(v) = e^{jv-(4v^2)}$, respectively. Let $Z = X - Y$.

a) Find $\overline{Z} = E[Z]$;

b) Find $\overline{Z^2} = E[Z^2]$;

c) Find the characteristic function of Z, $\Phi_Z(w)$.

$$\Phi_X(u) = e^{ju3-(2u^2)} = e^{ju\overline{X}-(\sigma_X^2 u^2/2)} \quad \therefore \quad \overline{X} = 3,\ \sigma_X^2 = 4, \text{ and } \overline{X^2} = \sigma_X^2 + \overline{X}^2 = 13$$

$$\Phi_Y(v) = e^{jv-(4v^2)} = e^{ju\overline{Y}-(\sigma_Y^2 v^2/2)} \quad \therefore \quad \overline{Y} = 1,\ \sigma_Y^2 = 8, \text{ and } \overline{Y^2} = \sigma_Y^2 + \overline{Y}^2 = 9$$

a) $\overline{Z} = E[Z] = E[X-Y] = E[X] - E[Y] = 3 - 1 = 2$

b) $\overline{Z^2} = E[Z^2] = E[(X-Y)^2] = E[X^2] + E[Y^2] - 2E[X]E[Y] = 13 + 9 - 2\times 3\times 1 = 22 - 6 = 16$

c) $E[-Y] = -E[Y] = -1,\ \ E[Y^2] = E[(-Y)^2] = 9$, and $\sigma_{-Y}^2 = 9 - (-1)^2 = 8 = \sigma_Y^2$

$$\Phi_{-Y}(v) = e^{jv(\overline{-Y})-(\sigma_{-Y}^2 v^2/2)} = e^{-jv-(4v^2)}$$

$$Z = X - Y = X + (-Y) \qquad\qquad f_Z(z) = f_X(z) * f_{-Y}(z)$$

Therefore, $\Phi_Z(w) = \Phi_X(w)\Phi_{-Y}(w) = e^{jw3-(2w^2)} e^{-jw-(4w^2)} = e^{jw2-(6w^2)}$

Indeed, $\sigma_Z^2 = \overline{Z^2} - (\overline{Z})^2 = 16 - 2^2 = 12$ and $(\sigma_Z^2/2) = 6$ which appears in the characteristic function.

Problem #3 (7 pts ea., 42 pts total)

Consider a signal/process $X(t) = A\cos(2\pi t + \Theta) + V(t)$ where A is a Gaussian random variable with zero mean and a variance of 8, Θ is uniformly distributed in $(0,2\pi)$ and the independent noise $V(t)$ has an autocorrelation function of $R_V(\tau) = 10e^{-1000|\tau|}$. The signal is applied to a finite-time integrator with impulse response $h(t) = [u(t) - u(t-2)]/2$.

a) Find the autocorrelation of the input $R_X(\tau)$;
b) Find the spectral density of the input $S_X(\omega)$;
c) Sketch the input spectral density;
d) Find the autocorrelation of the output $R_Y(\tau)$;
e) Find the spectral density of the output $S_Y(\omega)$;
f) Sketch the output spectral density.

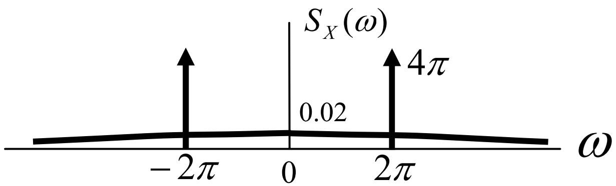
a) $$R_X(\tau) = E[X(t)X(t+\tau)] = E\{[A\cos(2\pi t + \Theta) + V(t)][A\cos(2\pi t + 2\pi\tau + \Theta) + V(t+\tau)]\}$$

$$= E\{A\cos(2\pi t + \Theta)A\cos(2\pi t + 2\pi\tau + \Theta)\} + R_V(\tau) = \frac{1}{2}E\{A^2\}\cos 2\pi\tau + R_V(\tau) = 4\cos 2\pi\tau + 10e^{-1000|\tau|}$$

b) $$S_X(\omega) = \mathbf{F}\{R_X(\tau)\} = \mathbf{F}\{4\cos 2\pi\tau\} + \mathbf{F}\{R_V(\tau)\}$$

$$= 4\pi\{\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\} + \frac{2\times 10^4}{10^6 + \omega^2}$$

c)



$$|H(\omega)|^2 = \text{sinc}^2\left(\frac{\omega}{\pi}\right)$$

$$Y(t) = \int_{-\infty}^{\infty} X(t-\lambda)h(\lambda)d\lambda = \frac{1}{2}\int_0^2 [A\cos(2\pi t - 2\pi\lambda + \Theta) + V(t-\lambda)]d\lambda$$

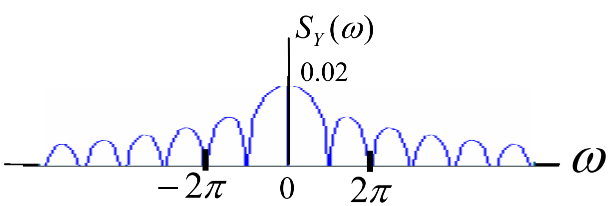
$$= \frac{1}{20\pi}A\sin(2\pi t - 2\pi\lambda + \Theta)\Big|_{\lambda=0}^{\lambda=2} + \left\{\frac{1}{2}\int_0^2 V(t-\lambda)d\lambda\right\} = \frac{1}{2}\int_0^2 V(t-\lambda)d\lambda = V(t) * h(t)$$

e) $$S_Y(\omega) = S_X(\omega)|H(\omega)|^2 = \text{sinc}^2\left(\frac{\omega}{\pi}\right)\left\{4\pi\{\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\} + \frac{2\times 10^4}{10^6 + \omega^2}\right\} = \text{sinc}^2\left(\frac{\omega}{\pi}\right)\frac{2\times 10^4}{10^6 + \omega^2} \approx 0.02\times \text{sinc}^2\left(\frac{\omega}{\pi}\right)$$

d) $$R_Y(\tau) = \mathbf{F}^{-1}\{S_Y(\omega)\} \approx \mathbf{F}^{-1}\left\{0.02\text{sinc}^2\left(\frac{\omega}{\pi}\right)\right\}$$

$$= 0.02\times\frac{1}{2}\left(1 - \frac{|\tau|}{2}\right) = 0.01\left(1 - \frac{|\tau|}{2}\right), \quad |\tau| < 2$$

f)



Problem #4 (7 pts ea., 21 pts total)
A random process having an autocorrelation of $R_X(\tau) = 16e^{-2|\tau|} + 16$ is the input to a linear system having an impulse response of $h(t) = \delta(t) - 2e^{-2t}u(t)$

a) Find the mean value of the output;
b) Find the mean-square value of the output;
c) Find the variance of the output.

a) $R_X(\tau) = 16e^{-2|\tau|} + 16$ The constant component in $R_X(\tau)$ corresponds to $(\overline{X})^2$. Therefore, $\overline{X} = 4$.

$$\overline{Y} = \overline{X}\int_{-\infty}^{\infty} h(t)dt = 4\int_0^{\infty}[\delta(t) - 2e^{-2t}]dt = 4[1-1] = 0$$

b) $$\overline{Y^2} = R_Y(0) = \int_0^{\infty} d\lambda_1 \int_0^{\infty} R_X(\lambda_2 - \lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 = \int_0^{\infty} h(\lambda_2)d\lambda_2 \int_0^{\infty} R_X(\lambda_2 - \lambda_1)h(\lambda_1)d\lambda_1$$

$$\int_0^{\infty} R_X(\lambda_2 - \lambda_1)h(\lambda_1)d\lambda_1 = \int_0^{\infty}[16e^{-2|\lambda_2 - \lambda_1|} + 16][\delta(\lambda_1) - 2e^{-2\lambda_1}]d\lambda_1 = \int_0^{\infty} 16e^{-2|\lambda_2 - \lambda_1|}[\delta(\lambda_1) - 2e^{-2\lambda_1}]d\lambda_1 = 16e^{-2|\lambda_2|} - \int_0^{\infty}[32e^{-2|\lambda_2 - \lambda_1|}e^{-2\lambda_1}]d\lambda_1$$

$$\int_0^{\infty}[32e^{-2|\lambda_2 - \lambda_1|}e^{-2\lambda_1}]d\lambda_1 = \int_0^{\lambda_2}[32e^{-2(\lambda_2 - \lambda_1)}e^{-2\lambda_1}]d\lambda_1 + \int_{\lambda_2}^{\infty}[32e^{-2(-\lambda_2 + \lambda_1)}e^{-2\lambda_1}]d\lambda_1 = 8e^{-2\lambda_2} + 32\lambda_2 e^{-2\lambda_2}$$

$$\int_0^{\infty} R_X(\lambda_2 - \lambda_1)h(\lambda_1)d\lambda_1 = 8e^{-2\lambda_2} - 32\lambda_2 e^{-2\lambda_2} \quad \text{for } \lambda_2 \ge 0$$

$$\overline{Y^2} = \int_0^{\infty}[8e^{-2\lambda_2} - 32\lambda_2 e^{-2\lambda_2}][\delta(\lambda_2) - 2e^{-2\lambda_2}]d\lambda_2 = 8 - \int_0^{\infty}[16e^{-4\lambda_2} - 64\lambda_2 e^{-4\lambda_2}]d\lambda_2 = 8 - 4 + \int_0^{\infty} 64\lambda_2 e^{-4\lambda_2}d\lambda_2 = 4 + 4 = 8$$

Alternatively,

$$S_X(\omega) = \mathbf{F}\{R_X(\omega)\} = 16 \times 2\pi\delta(\omega) + 16\frac{4}{4+\omega^2} = 32\pi\delta(\omega) + \frac{64}{4+\omega^2}$$

$$\mathbf{F}\{h(t)\} = \mathbf{F}\{\delta(t) - 2e^{-2t}u(t)\} = 1 - \frac{2}{2+j\omega} = \frac{j\omega}{2+j\omega} = H(\omega) \qquad \therefore \quad |H(\omega)|^2 = \frac{\omega^2}{4+\omega^2}$$

$$S_Y(\omega) = S_X(\omega)|H(\omega)|^2 = \left[32\pi\delta(\omega) + \frac{64}{4+\omega^2}\right]\frac{\omega^2}{4+\omega^2} = \frac{64\omega^2}{(4+\omega^2)^2}$$

$$\overline{Y^2} = R_Y(0) = \frac{1}{\pi}\int_0^{\infty}\frac{64\omega^2}{(4+\omega^2)^2}d\omega = \frac{64}{\pi}\left[\frac{-\omega}{2(4+\omega^2)} + \frac{1}{4}\tan^{-1}\left(\frac{\omega}{2}\right)\Bigg|_0^{\infty}\right] = \frac{64}{\pi}\frac{\pi}{8} = 8$$

c) $\sigma_Y^2 = \overline{Y^2} - (\overline{Y})^2 = 8$

