H.W #9 Solutions

$\boxed{\text{Prob 9.1}}$ (7-3.3)

$$S_X(\omega) = 32\pi\,\delta(\omega) + 8\pi\,\delta(\omega\pm 6) + 32\pi\,\delta(\omega\pm 12)$$

(a)

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, e^{j\omega\tau}\, d\omega$$

$$E\{X^2(t)\} = R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$$

$$= \frac{1}{2\pi}[\,32\pi\times 3 + 8\pi\times 2\,]$$

$$= 16\times 3 + 4\times 2 = \boxed{56}$$

(b)

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(\omega)\}$$

$$= 16 + 8\cos(6\omega) + 16\cos(12\omega)$$

$$R_X'(\tau) = 16 \qquad \therefore \boxed{E\{X(t)\}} = \boxed{\pm 4}$$

$$E\{X(t)\}^2 = \lim_{\tau\to\infty} R_X'(\tau) = 16$$

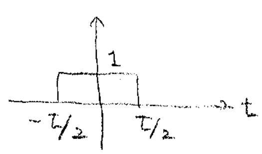
$$\boxed{Var\{X(t)\}} = E\{X^2(t)\} - E\{X(t)\}^2 = \boxed{40}$$

(c)

$$\boxed{\omega} = \boxed{0,\ \pm 6,\ \pm 12}$$

$\boxed{\text{Prob 9.2}}$ (7-3.4)

Let $rect(t/\tau) \overset{def}{=} \begin{cases} 1 & |t|\le \tau \\ 0 & o.w \end{cases}$



then,

$$X(t) = \sum_{n=-\infty}^{\infty} A_n\, rect\left[\frac{t-(t_0+0.1\cdot n)}{0.05}\right]$$

, where

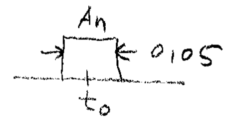
$$\begin{cases} P[A_n = 2] = 1/2 \\ P[A_n = 0] = 1/2 . \end{cases}$$

$t_0 \sim u[0, 0.1]$

$$\underset{\langle A_n, t_0\rangle}{E}\{X(t)\} = \underset{\langle t_0\rangle}{E}\{\underset{\langle A_n\rangle}{E}\{X(t)\,|\,t_0\}\}$$

$$\underset{\langle A_n\rangle}{E}\{X(t)\,|\,t_0\} = \begin{cases} \underset{\langle A_n\rangle}{E}\{A_n\} & t\in \text{pulse} \\ 0 & t\notin \text{pulse} \end{cases}$$

A pulse with height A_n and width 0.05 is located at $t = t_0$.



For any $t\in[0, 0.1]$, the probability $P[t\in \text{pulse}] = 1/2$ and $P[t\notin \text{pulse}] = 1/2$.

$$\therefore \underset{\langle A_n\rangle}{E}\{X(t)\,|\,t_0\} = \begin{cases} \underset{\langle A_n\rangle}{E}\{A_n\} = 1 & t\in \text{pulse} \\ & (\text{Prob} = 1/2) \\ 0 & t\notin \text{pulse} \\ & (\text{Prob} = 1/2) \end{cases}$$

$$\therefore \boxed{\underset{A_n, t_0}{E}\{X(t)\}} = \underset{\langle t_0\rangle}{E}\{E\{X(t)\,|\,t_0\}\}$$

$$= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 = \boxed{\frac{1}{2}}$$

Similary

$$\underset{\langle A_n\rangle}{E}\{X^2(t)\,|\,t_0\} = \begin{cases} E\{A_n^2\} = 2 & t\in\text{pulse} \\ 0 & t\notin\text{pulse} \end{cases}$$

$$\therefore \boxed{E\{X^2(t)\}} = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 0 = \boxed{1}$$

$$\boxed{Var\{X(t)\}} = E\{X^2(t)\} - E\{X(t)\}^2 = 1 + \frac{1}{2}^2 = \boxed{\frac{3}{4}}$$

(c) Use (n-25)

$$F_1(\omega) = \mathcal{F}\{\operatorname{rect}(t/\tfrac{1}{20})\} = \frac{1}{20}\,Sa\left(\frac{\omega}{40}\right)$$

$$|F_1(\omega)|^2 = \frac{1}{400}\,Sa^2\left(\frac{\omega}{40}\right)$$

$$S_X(\omega) = |F_1(\omega)|^2\left[\frac{\sigma_Y^2}{t_1} + \frac{2\pi\,(\bar{Y})^2}{t_1^2}\sum_{n=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi}{t_1}n\right)\right]$$

$\sigma_Y^2 =$ $\quad t_1 = 0.1, \quad \bar{Y} = 1$

$$\bar{Y} = \frac{1}{2}[0+2] = 1$$

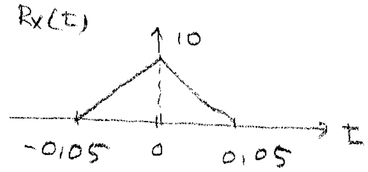
$$\overline{Y^2} = \frac{1}{2}[0+2^2] = 2$$

$$\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = 2 - 1 = 1.$$

$$= \boxed{2.5\times10^{-3}\,Sa^2\left(\frac{\omega}{40}\right)\left[10 + 200\pi\sum_{n=-\infty}^{\infty}\delta(\omega - 20\pi n)\right]}$$

$\boxed{\text{Prob 9.3}}$ (7-6.1)

$$R_X(\tau) = \begin{cases} 10\left[1 - \dfrac{|\tau|}{0.05}\right] & |\tau| \le 0.05 \\ 0 & \text{o.w.} \end{cases}$$



(a) $\boxed{E\{X(t)\}} = \sqrt{\lim_{\tau\to\infty} R_X(\tau)} = \boxed{0}$

$\boxed{E\{X^2(t)\}} = R_X(0) = \boxed{10}$

$\boxed{\mathrm{Var}\{X(t)\}} = E\{X^2(t)\} - E\{X(t)\}^2 = \boxed{10}$

(b) $\boxed{S_X(\omega)} = \mathcal{F}\{R_X(\tau)\}$

$= 200\,\mathcal{F}\{\mathrm{rect}(\tau/0.05) * \mathrm{rect}(\tau/0.05)\}$

$= 200\,\mathcal{F}\{\mathrm{rect}(t/0.05)\}^2$

$= 200\left[0.05\,Sa\left(\dfrac{\omega \cdot 0.05}{2}\right)\right]^2$

$= \boxed{\dfrac{1}{2}\,Sa\left(\dfrac{\omega}{40}\right)^2}$ or

$= \boxed{\dfrac{1}{2}\left[\dfrac{\sin^2\left(\frac{\omega}{40}\right)}{\left(\frac{\omega}{40}\right)^2}\right]}$

(c) At $\tau = \tau_1 = 0.05$, $R_X(\tau_1) = 0$.

At $f = f_1 = 20$,

$S_X(f_1) = 0$,

$20 = \boxed{f_1 = 1/\tau_1} = 1/0.05$

$\boxed{\text{Prob 9.4}}$ (7-6.2)

$R_X(\tau) = 16 e^{-5|\tau|} \cos 20\pi\tau + 8 \cos 10\pi\tau$

(a) $R_X'(\tau) = 16 e^{-5|\tau|} \cos 20\pi\tau$

$\lim_{\tau\to\infty} R_X'(\tau) = 0 = E\{X(t)\}^2$

$\therefore E\{X(t)\} = 0$.

$E\{X^2(t)\} = R_X(0) = 16 + 8 = 24$.

$\therefore \boxed{\mathrm{Var}\{X(t)\}} = 24 - 0 = \boxed{24}$

(b) $S_X(\omega) = \mathcal{F}\{R_X(\tau)\}$

$R_X(\tau) = 16 e^{-5|\tau|} \cdot \frac{1}{2}\left[e^{j20\pi\tau} + e^{-j20\pi\tau}\right] + 4\left[e^{j10\pi\tau} + e^{-j10\pi\tau}\right]$

since $e^{-\alpha|\tau|} \leftrightarrow \dfrac{2\alpha}{\alpha^2 + \omega^2}$,

$f(t)e^{j\omega_0 t} \leftrightarrow F(\omega - \omega_0)$

$$\boxed{S_X(\omega)} = \boxed{16 \cdot \left[\frac{5}{25 + (\omega + 20\pi)^2} + \frac{5}{25 + (\omega - 20\pi)^2}\right] + 8\pi\left[\delta(\omega + 10\pi) + \delta(\omega - 10\pi)\right]}$$

(c) $\boxed{S_X(0)} = 16\left[\dfrac{5}{25 + 400\pi^2} + \dfrac{5}{25 + 400\pi^2}\right] = \boxed{0.0403}$

$\boxed{\text{Prob 9.5}}$ (7-8.1)

$$S_X(\omega) = \frac{16}{\omega^2+16} \qquad S_Y(\omega) = \frac{\omega^2}{\omega^2+16}$$

$U(t) = X(t) + Y(t)$ $\qquad X(t), Y(t)$ indep.

(a) $R_U(\tau) = R_X(\tau) + R_Y(\tau)$ $\quad (\because X, Y \text{ indep})$

$\Rightarrow \boxed{S_U(\omega)} = S_X(\omega) + S_Y(\omega) = \boxed{1}$

(b) $R_{XY}(\tau) = E\{X(t+\tau)Y(t)\}$

$= \cancel{E\{X(t+\tau)\}}^{\,0}\, E\{Y(t)\}$

$E\{X(t)\} = \pm\sqrt{\lim_{\tau\to\infty} R_X(\tau)} = 0, \quad \because R_X(\tau) = 2\cdot e^{-4|\tau|} \leftrightarrow \dfrac{2\cdot 8}{\omega^2+16}$

$\therefore R_{XY}(\tau) = 0 \quad \Rightarrow \boxed{S_{XY}(\omega)} = \mathcal{F}\{R_{XY}(\tau)\} = \boxed{0}$

(c) $R_{XU}(\tau) = E\{X(t+\tau)U(t)\}$

$= E\{X(t+\tau)[X(t)+Y(t)]\}$

$= E\{X(t+\tau)X(t)\} + E\{X(t+\tau)Y(t)\}$

$= R_X(\tau) + \cancel{E\{X(t+\tau)\}}^{\,0}\, E\{Y(t)\}$

$\therefore \boxed{S_{XU}(\omega)} = \mathcal{F}\{R_{XU}(\tau)\} = \mathcal{F}\{R_X(\tau)\} = S_X(\omega) = \boxed{\dfrac{16}{\omega^2+16}}$

$\boxed{\text{Prob } 9.6}$ (n-11, 1)

$$f(t) = \begin{cases} \frac{1}{2}[1+\cos(\frac{\pi}{t_1} t)] & |t| \le t_1 \\ 0 & \text{o.w.} \end{cases}$$

(a)

$$= \frac{1}{2} \text{rect}(t/2t_1) [1+\cos(\frac{\pi}{t_1} \cdot t)]$$

use the properties

$$\text{rect}(t/\tau) \overset{\mathcal{F}}{\leftrightarrow} \tau Sa(\frac{\omega \tau}{2}),$$

$$f(t) \cdot \cos(\omega_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2} F(\omega+\omega_0) + \frac{1}{2} F(\omega-\omega_0)$$

$$\therefore F(\omega) = \mathcal{F}\{f(t)\}$$

$$= t_1 Sa(\omega t_1) + \frac{t_1}{2} Sa[(\omega + \frac{\pi}{t_1}) t_1] + \frac{t_1}{2} Sa[(\omega - \frac{\pi}{t_1}) t_1]$$

$$= t_1 \frac{\sin(\omega t_1)}{(\omega t_1)} + \frac{t_1}{2} \cdot \frac{\sin(\omega t_1 + \pi)}{(\omega t_1 + \pi)} + \frac{t_1}{2} \frac{\sin(\omega t_1 - \pi)}{(\omega t_1 - \pi)}$$

$$= t_1 \frac{\sin \omega t_1}{\omega t_1} - \frac{t_1}{2} \cdot \frac{\sin(\omega t_1)(2\omega t_1)}{\omega^2 t_1^2 - \pi^2}$$

$$= t_1 \frac{\sin \omega t_1}{\omega t_1} [1 - \frac{\omega^2 t_1^2}{\omega^2 t_1^2 - \pi^2}]$$

$$= t_1 \cdot \frac{\sin \omega t_1}{\omega t_1} \cdot \frac{(-\pi^2)}{\omega^2 t_1^2 - \pi^2}$$

$$\boxed{S_X(\omega)} = |F(\omega)|^2$$

$$= \boxed{t_1^2 \left[\frac{\sin \omega t_1}{\omega t_1}\right]^2 \cdot \left[\frac{\pi^2}{\omega^2 t_1^2 - \pi^2}\right]^2}$$

(b)

$$\frac{S_X(\omega_1)}{\max\{S_X(\omega)\}} = \left[\frac{\sin \omega_1 t_1}{\omega_1 t_1}\right]^2 \left[\frac{\pi^2}{\omega_1^2 t_1^2 - \pi^2}\right]^2$$

$$= 0.01.$$

$$\Rightarrow \omega_1 t_1 \approx 5.18$$

$$\Rightarrow \boxed{\omega_1} \approx \boxed{\frac{5.18}{t_1}}$$

(c) The duration of a pulse is inversely proportional to the bandwidth.