ECE-3075A Random Signals

Fall 2003 – Quiz #3

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Problem 1 (15 pts each)

A random process, defined by $X(t) = A + B\cos(10\pi t + \Theta)$ where Θ, A and B are independent random variables, Θ is uniformly distributed in (0, 2π), A is a Gaussian random variable with a mean value of 4 and a variance of 9, and B is uniformly distributed in (-3,+3).

a. Find the expected value of $X(t)$;
b. Find the autocorrelation function of the random process $X(t)$;
c. Determine if the process is wide sense stationary; provide justification.

Answer:

a. $E[X(t)] = E[A + B\cos(10\pi t + \Theta)] = E[A] + E[B]E[\cos(10\pi t + \Theta)] = E[A] = 4$

b. Note that $E[A] = \overline{A} = 4$ and $\sigma_A^2 = \overline{A^2} - (\overline{A})^2 = 9$, thus $\overline{A^2} = 9 + 4^2 = 25$. Also, $\overline{B} = 0$ and $\overline{B^2} = 6^2/12 = 3$.

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$
$$= E[A^2 + AB\cos(10\pi t + 10\pi\tau + \Theta) + AB\cos(10\pi t + \Theta) + B^2\cos(10\pi t + \Theta)\cos(10\pi t + 10\pi\tau + \Theta)]$$
$$= \overline{A^2} + 0 + 0 + \overline{B^2}E[\cos(10\pi t + \Theta)\cos(10\pi t + 10\pi\tau + \Theta)] = 25 + \frac{3}{2}\cos 10\pi\tau = R_{XX}(\tau)$$

c. Since the expected value is independent of time and the autocorrelation function $R_{XX}(t_1, t_2)$ only depends on the time difference between t_1 and t_2, $\tau = t_1 - t_2$, the process is wide-sense stationary.

Problem 2 (15 pts each)

Consider a process $Y(t) = X(t) + V(t)$ where $X(t) = (1 + A)\exp(-t)$ is considered the signal (something useful) with A being a Gaussian random variable with zero mean and a variance of 11, and $V(t)$ is an independent noise process, time samples of which are independently and identically distributed zero mean Gaussian random variables with a variance of 3.

a. Find the expected value of $Y(t)$;
b. Find the autocorrelation function of $Y(t)$;
c. If we define instantaneous power to be $E[Y^2(t)], E[X^2(t)]$ and $E[V^2(t)]$, determine the time at which the signal and the noise have equal instantaneous power.

Answer:

a. $E[X(t)] = E[(1+A)\exp(-t)] = (1+E[A])\exp(-t) = \exp(-t)$ and $E[V(t)] = 0$

Therefore, $E[Y(t)] = E[X(t)] + E[V(t)] = \exp(-t)$

b. Note that $E[A] = \overline{A} = 0$ and $\sigma_A^2 = \overline{A^2} - (\overline{A})^2 = 11$, thus $\overline{A^2} = 11$.

$R_Y(t, t+\tau) = E[Y(t)Y(t+\tau)] = E\{[(1+A)\exp(-t) + V(t)][(1+A)\exp(-t-\tau) + V(t+\tau)]$

$= E[(1+2A+A^2)\exp(-t)\exp(-t-\tau)] + E[V(t)V(t+\tau)]$ ← - - - - Since $E[V(t)] = 0$, the cross terms in the equation vanish.

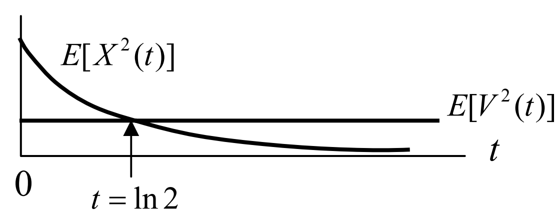
$= E[1+2A+A^2]\exp(-2t-\tau) + E[V(t)V(t+\tau)]$

$= (1+2\times 0+11)\exp(-2t-\tau) + E[V(t)V(t+\tau)] = 12\exp(-2t-\tau) + 3\delta(\tau) = R_X(t, t+\tau) + R_V(\tau)$

c. $E[X^2(t)] = R_X(t,t) = 12\exp(-2t)$

$E[V^2(t)] = R_V(0) = 3$

$3 = 12\exp(-2t) \quad \Rightarrow \quad t = \ln 2$



Problem 3 (15 pts)

As discussed in class, we are interested in using hypothesis testing to determine if a signal *x,* which has a value of 4, is present in a noisy observation *y*. We assume noise *v* to be additive, i.e., $y = x+v$, and Gaussian with zero mean and variance $\sigma_v^2 = 1$. We form the hypothesis as follows:

H_0: Signal *x* is present in the noisy observation *y*, $f_Y(y\,|\,H_0) = \frac{1}{\sqrt{2\pi}\sigma_v}\exp\left\{-\frac{(y-x)^2}{2\sigma_v^2}\right\}$

H_1: Signal *x* is not present in the noisy observation *y*, $f_Y(y\,|\,H_1) = \frac{1}{\sqrt{2\pi}\sigma_v}\exp\left\{-\frac{y^2}{2\sigma_v^2}\right\}$

We perform a log likelihood ratio test upon observation of y_0 to accept H_0 if $\log\frac{f_Y(y_0\,|\,H_0)}{f_Y(y_0\,|\,H_1)} \ge \lambda$ and to reject H_0 if $\log\frac{f_Y(y_0\,|\,H_0)}{f_Y(y_0\,|\,H_1)} < \lambda$. We also assume that the hypotheses are equally likely a priori. If the testing threshold λ is chosen such that the probability of "false alarm" is 0.1587, what would the probability of "miss" be? (Hint: use the table of normal distribution, Appendix D)

Answer:

let $\gamma = e^{\lambda}$. False alarm occurs if $f_Y(y_0\,|\,H_1) > \frac{1}{\gamma} f_Y(y_0\,|\,H_0)$. The probability of false alarm is given as 0.1587, which translates to the choice of 1 as the corresponding threshold in terms of the value of the observation, *y*.

That is to say, if we observe a value of *y*, $y > 1$, we'd accept null hypothesis and declare that the signal is present. The probability that the signal was not there but $y > 1$ is 0.1587. Then the probability of "miss" is

$$\Phi(1-4) = \Phi(-3) = 1 - \Phi(3) = 1 - 0.9987 = 0.0013$$

due to the fact that the mean is shifted to x=4 and $\sigma_v^2 = 1$.