Problem 1 (8 pts. each, 40 total)

A random process, defined by $X(t) = A\exp(-\beta t), \quad t \geq 0$ where β is a constant and A is a Gaussian random variable with a mean value of 4 and variance of 9. A sample function of the random process is observed to have the following values: $x(1) = 1.3534$ and $x(3) = 0.24878 \times 10^{-1}$.

a. Find the value of A and β in the sample function of time;
b. Is this a deterministic process or a non-deterministic process? State your reason to validate your answer.
c. Find the expected value of $X(t)$;
d. Find the autocorrelation function of the random process;
e. Determine if the process is stationary in any sense.

a. $x(1) = 1.3534 = A\exp(-\beta); \quad x(3) = 0.24788 \times 10^{-1} = A\exp(-3\beta)$

$x(1)/x(3) = \exp(2\beta) = 54.6 \quad \Rightarrow \quad \beta = 0.5\ln(54.6) = 2, \quad A = 1.3534\exp(2) = 10$

b. It is deterministic because it, as a function of time, is entirely specified once A is realized.

c. $E[X(t)] = E[A\exp(-\beta t)] = E[A]\exp(-\beta t) = 4\exp(-2t)$

d. Note that $E[A] = \overline{A} = 4$ and $\sigma_A^2 = \overline{A^2} - (\overline{A})^2 = 9$, thus $\overline{A^2} = 9 + 4^2 = 25$.

$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = E[A^2\exp(-2t_1)\exp(-2t_2)] = E[A^2]\exp[-2(t_1 + t_2)] = 25\exp[-2(t_1 + t_2)]$

e. Since the expected value is a function of time and the autocorrelation function $R_{XX}(t_1, t_2)$ depends on the origin of time, the process is not stationary.

Problem 2 (12 pts each, 36 total)

Consider a wide sense stationary process $X(t) = (1+A)\cos(\omega t + \Theta) + V(t)$ where A is a Gaussian random variable with zero mean and a variance of 11, Θ is an independent random variable uniformly distributed over (0, 2π), and $V(t)$ is an independent noise process, time samples of which are independently and identically distributed zero mean Gaussian random variables with a variance of 3.

a. Find the expected value of $X(t)$;
b. Find the autocorrelation function of $X(t)$;
c. If we consider the sinusoidal portion of the process the signal and $V(t)$, as stated, the noise, determine the signal-to-noise ratio (SNR) of $X(t)$.

a. $E[X(t)] = E[(1+A)\cos(\omega t+\Theta)] + E[V(t)] = E[1+A]E[\cos(\omega t+\Theta)] + E[V(t)] = 0$

because $V(t)$ is an independent zero-mean process, and

$$E[\cos(\omega t+\Theta)] = \int_0^{2\pi} \cos(\omega t+\theta)\frac{1}{2\pi}d\theta = \frac{\sin(\omega t+\theta)}{2\pi}\Bigg|_{\theta=0}^{2\pi} = \frac{\sin(\omega t+2\pi)-\sin(\omega t)}{2\pi} = 0$$

b. Note that $E[A] = \overline{A} = 0$ and $\sigma_A^2 = \overline{A^2} - (\overline{A})^2 = 11$, thus $\overline{A^2} = 11$.

$$E[X(t)X(t+\tau)] = E\{[(1+A)\cos(\omega t+\Theta)+V(t)][(1+A)\cos(\omega t+\omega\tau+\Theta)+V(t+\tau)]$$

$$= E[(1+2A+A^2)\cos(\omega t+\Theta)\cos(\omega t+\omega\tau+\Theta)] + E[V(t)V(t+\tau)]$$

← - - - - Since $E[V(t)]=0$, the cross terms in the equation vanish.

$$= E[1+2A+A^2]E[\cos(\omega t+\Theta)\cos(\omega t+\omega\tau+\Theta)] + E[V(t)V(t+\tau)]$$

$$= (1+2\times 0+11)\left[\frac{1}{2}E\{\cos\omega\tau + \cos(2\omega t+\omega\tau+2\Theta)\}\right] + E[V(t)V(t+\tau)] = 6\cos\omega\tau + 3\delta(\tau)$$

c. The sinusoidal portion has a mean square value of 6 (set τ=0) and the noise portion has a mean square value of 3. Therefore,

$$SNR = 10\log_{10}\frac{6}{3} = 10\log_{10}2 = 3.0103\text{ dB}$$

Problem 3 (8 pts each, 24 total)

A finite-state Markov chain is characterized by the following specifications:

Initial state probability vector: $[0.5 \quad 0.4 \quad a]^t$; State transition probability matrix: $\begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.2 & b & 0.1 \\ 0.1 & c & 0.6 \end{bmatrix}$

a. Find the value of a, b and c;

b. Enumerate the probability for the state sequence: (3,2,1,2,1,2,2,3,3,1,1);

c. Of all the state sequences that are more than 5 in length, e.g. (1,2,2,3,1,2,…), which state sequence has the highest probability?

a. $0.5+0.4+a=1,$ therefore $a=0.1;$ $0.2+b+0.1=1,$ therefore $b=0.7;$ $0.1+c+0.6=1,$ therefore $c=0.3.$

b. $\Pr\{(3,2,1,2,1,2,2,3,3,1,1)\} = 0.1\times0.3\times0.2\times0.4\times0.2\times0.4\times0.7\times0.1\times0.6\times0.1\times0.6 = 0.4838\times10^{-6}$

c. $a=0.7$ is the highest probability entry in the state transisiton probability matrix. It is counted when the system continues to stay in state 2. The most probable sequence would thus look like (?,?,2,2,2,2,...). The first few states need closer comparison. The most competitive one is (1,2,2,2,2,..) due to the fact that state 1 has the highest initial state probability of 0.5. However, the transition from state 1 to state 2 has a probability of 0.4, resulting in a chain of probabilities that looks like $0.5\times0.4\times0.7\times0.7\times....$ The sequence (2,2,2,2,2,...) has the chain of probabilities: $0.4\times0.7\times0.7\times0.7\times....$ As a result, we can conclude that the most probable sequence is (2,2,2,2,2,2,....).