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ECE 3075

HW #5

Solution

Problem 5.1 (2-6.2)

(a) $Y = g(X) = X^3$, $g'(x) = 3x^2$.

Solving x in terms of y, we get only one root:

$x_1 = y^{1/3}$, $-\infty < y < \infty$.

From the theorem,

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} = \frac{\frac{1}{\sqrt{2\pi}} e^{-(y^{1/3})^2/2}}{|3(y^{1/3})^2|} = \boxed{\frac{1}{3\sqrt{2\pi}} y^{-2/3} e^{-y^{2/3}/2}}$$

(b) $$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)dy = \frac{1}{3\sqrt{2\pi}} \int_{-\infty}^{\infty} y^{1/3} e^{-\frac{y^{2/3}}{2}} dy$$

Note that $y^{1/3} e^{-y^{2/3}/2}$ is an odd function $\Rightarrow$ its integral from $-\infty$ to ∞ is zero.

$\Rightarrow \boxed{E[Y] = 0}$

(c) $\mathrm{Var}[Y] = E[Y^2] - \underbrace{E^2[Y]}_{0} = E[Y^2] = E[X^6]$

From (2-27) in text, we have $E[(X-\bar{X})^6] = E[X^6] = 1.3.5\,\sigma^6$

$= 15.1^6$

$= \boxed{15}$

Problem 5.2 (2-6.6) Let Y_i, $i=1..5$ denote the independent Gaussian outcomes.

(a) $E[X^2] = E[\sum_{i=1}^{5} Y_i^2] = \sum_{i=1}^{5} E[Y_i^2] = \sum_{i=1}^{5}(\sigma_i^2 + \cancel{\mu_i^2}^{\,0}) = 5.\sigma^2 = \boxed{5}$ $(= n\sigma^2)$

(b) $\mathrm{Var}[X^2] = 2n\sigma^4 = 2.5.1^2 = \boxed{10}$

(c) The answer depends on the interpretation of "most probable value":

(Cont'd)

i) Value of y where $f_{X^2}(y)$ attains maximum (mode of the distribution)

ii) Expected value of X^2, which is $\int_{-\infty}^{\infty} y f_{X^2}(y)\,dy$ (mean of the distribution)

iii) Value of y where $\Pr\{\tilde{X}^2 < y\} = \Pr\{\tilde{X}^2 > y\}$, which is the solution of the eqn:

$$\int_{-\infty}^{y} f_{X^2}(z)\,dz = \int_{y}^{\infty} f_{X^2}(z)\,dz \quad \text{(median of the distribution)}$$

For interpretation (i), the answer is calculated as follows:

$$\frac{\partial}{\partial y} f_{X^2}(y) = \frac{\partial}{\partial y}\left[\frac{y^{3/2}}{2^{5/2}\,\Gamma(5/2)} e^{-y/2}\right] = \frac{3/2\, y^{1/2}}{\cancel{2^{5/2}\,\Gamma(5/2)}} \cancel{e^{-y/2}} - \frac{1}{2}\frac{y^{3/2}}{\cancel{2^{5/2}\Gamma(5/2)}}\cancel{e^{-y/2}} = 0$$

$$\Rightarrow \frac{3}{2} = \frac{y}{2} \Rightarrow \boxed{y=3}$$

For interpretation (ii), the answer is

$$E[X^2] = \boxed{5}$$

Interpretation (iii) does not yield a closed-form solution. It is solved numerically using the following relation for y:

$$F_Y(y) = 1 - e^{-y/2}\sum_{k=0}^{4}\frac{1}{k!}\left(\frac{y}{2}\right)^k = \frac{1}{2} \Rightarrow \boxed{e^{-y/2}\left(1+\frac{y}{2}+\frac{y^2}{8}+\frac{y^3}{48}+\frac{y^4}{384}\right) = \frac{1}{2}}$$

All three of the answers are correct. They all find usage in different applications where the optimization criterion is stated in different ways. For example, it can be shown that:

i) The mode maximizes the probability that the r.v. X^2 is within an interval of infinitesimal width:

$$\text{mode}(X^2) = \max_y \Pr\{y-\epsilon < \tilde{X}^2 < y+\epsilon\}, \quad \epsilon \text{ infinitesimally small.}$$

ii) The mean minimizes the average squared distance:

$$\text{mean}(X^2) = \min_y E[|y-\tilde{X}^2|^2]$$

iii) The median minimizes the average absolute distance:

$$\text{median}(X^2) = \min_y E[|y-\tilde{X}^2|]$$

Problem 5.3 (2-7.3) $f_t(t) = \frac{1}{6} e^{-t/6}, \; t \geq 0$

(a) $\int_0^6 f_t(t)\,dt = \frac{1}{6}\left(-6 e^{-t/6}\right)_0^6 = 1 - e^{-1} = \boxed{0{,}632}$

(b) $\int_{10}^{\infty} f_t(t)\,dt = \frac{1}{6}\left(-6 e^{-t/6}\right)_{10}^{\infty} = e^{-10/6} = \boxed{0{,}189}$

(c) $\int_5^6 f_t(t)\,dt = \frac{1}{6}\left(-6 e^{-t/6}\right)_5^6 = e^{-1} - e^{-6/5} = \boxed{0{,}0667}$

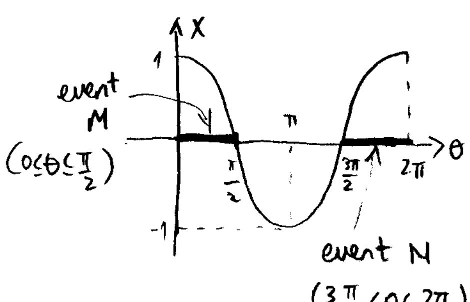
Problem 5.4 (2-8.2)

(a) From the continuous version of Bayes' Theorem:

$$f(x|M) = \frac{\Pr\{M \mid \tilde{x} = x\}}{\Pr\{M\}} f_x(x)$$

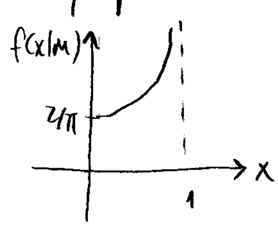
$$\Rightarrow f(x \mid 0 \leq \theta \leq \tfrac{\pi}{2}) = \frac{\Pr\{0 \leq \theta \leq \frac{\pi}{2} \mid \tilde{x} = x\}}{\Pr\{0 \leq \theta \leq \frac{\pi}{2}\}} \cdot \frac{1}{\pi\sqrt{1-x^2}} = \frac{\Pr\{0 \leq \theta \leq \frac{\pi}{2} \mid \tilde{x} = x\}}{\frac{\pi}{4}\sqrt{1-x^2}}$$

Let's examine the term $\Pr\{0 \leq \theta \leq \frac{\pi}{2} \mid \tilde{x} = x\}$:



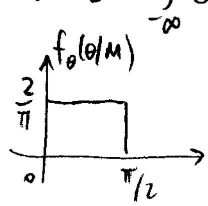
It is obvious that the above probability is zero when $-1 \leq x \leq 0$, since no θ between $0 \leq \theta \leq \frac{\pi}{2}$ yields a negative value.
When $0 \leq x \leq 1$, the probability that θ is between 0 and $\frac{\pi}{2}$ is equal to $1/2$, since event M and event N are equally probable because of the uniformity of θ.

$$\Rightarrow \boxed{f(x|M) = \begin{cases} \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}, & 0 \leq x \leq 1 \\ 0, & \text{else} \end{cases}}$$



(b) $E[X|M] = E[\cos\theta \mid M] = \int_{-\infty}^{\infty} \cos\theta \, f_\theta(\theta|M)\,d\theta$.

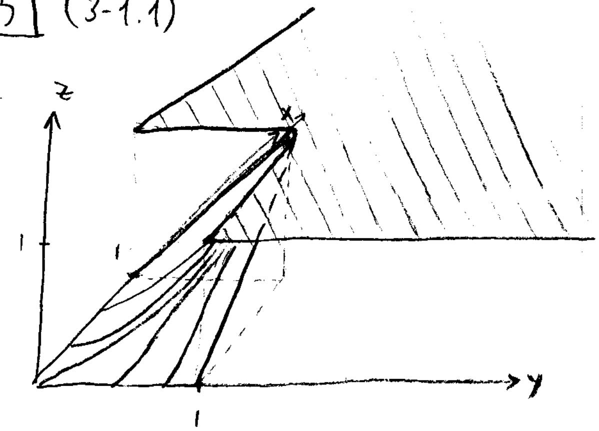
Since θ is uniform,



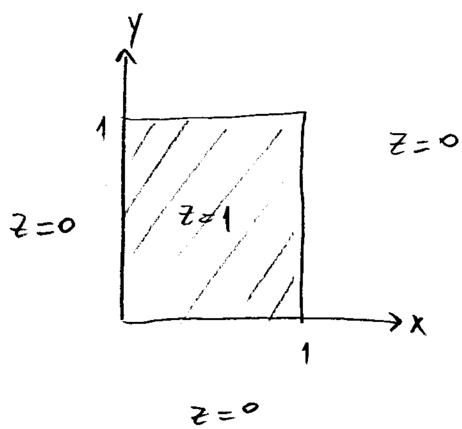
$\Rightarrow E(X|M) = \int_0^{\pi/2} \cos\theta \cdot \frac{2}{\pi}\,d\theta = \boxed{\frac{2}{\pi}}$

$\boxed{\text{Problem 5.5}}$ (3-1.1)

(a)



(b) $f(x,y) = \dfrac{\partial^2}{\partial x \partial y} F(x,y) = \begin{cases} 0, & x<0,\ y<0 \\ 1, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & x>1,\ y>0 \end{cases}$



(c) $\Pr\left\{ X \le \frac{3}{4},\ Y > \frac{1}{4} \right\} = \int_{x=0}^{3/4} \int_{y=1/4}^{1} dy\,dx = \left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = \boxed{\frac{9}{16}}$