Ilker R. Capoglu ECE 3075 10/01/2003

HW #6

Solution

Problem 6.1 (3-1.4)

(a) $\Pr\{X\le 3, Y>3\} = \Pr\{X\le 3\}\Pr\{Y>3\}$, since X & Y are independent.

$= \left(3\cdot\frac{1}{6}\right)\left(3\cdot\frac{1}{6}\right) = \boxed{0.25}$

(b) $E[XY] = E[X]E[Y] = E^2[X]$, since X & Y are indep. and identically distributed.

$E[X] = \sum_{x=1}^{6} x\, p(x) = \sum_{x=1}^{6} x\cdot\frac{1}{6} = \frac{1}{6}(1+2+\cdots 6) = 3.5$

$\Rightarrow E[XY] = \frac{49}{4} = \boxed{12.25}$

(c) $E[\frac{X}{Y}] = E[X]E[\frac{1}{Y}]$

$E[\frac{1}{Y}] = \sum_{y=1}^{6}\frac{1}{y}\, p(y) = \frac{1}{6}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{6}\right) \doteq 0{,}4083$

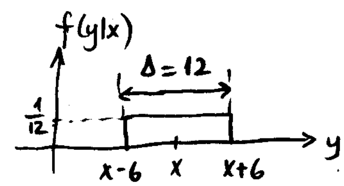
$\Rightarrow E[\frac{X}{Y}] = (3.5)(0{,}4083) = \boxed{1{,}429}$

Problem 6.2 (3-2.1) $f_X(x) = \frac{x}{\sigma^2}e^{-x^2/2\sigma^2}$, $\bar{X} = \sqrt{\frac{\pi}{2}}\,\sigma = 10 \Rightarrow \sigma = 10\sqrt{\frac{2}{\pi}} = 7{,}979$

(a) $f(x|y) = \dfrac{f(y|x)f(x)}{\int_{-\infty}^{\infty} f(y|x)f(x)dx}$

Since $Y = X+N$, when X is given, Y becomes a uniform r.v. with mean X and variance 12. From (2-39) in textbook,

$\sigma^2_{Y|X} = \sigma^2_N = \frac{(x_2-x_1)^2}{12} = 12 \Rightarrow \Delta = x_2 - x_1 = 12$:



$f(y|x) = \frac{1}{12}\left[u(y-(x-6)) - u(y-(x+6))\right]$, where $u(x)$ is the unit step function

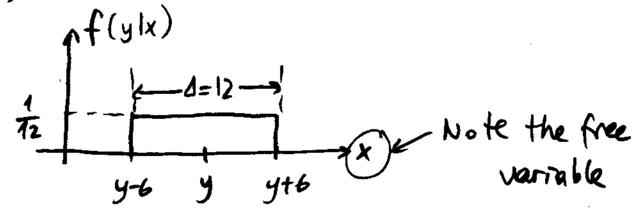
$\Rightarrow$

We want to express $f(y|x)$ with x as the free variable, because we will take the integral in the denominator

$$\int_{-\infty}^{\infty} f(y|x) f(x) dx$$

with respect to x. Noting that $u(-x) = 1 - u(x)$, we obtain

$$f(y|x) = \frac{1}{12}\left[u(x-(y-6)) - u(x-(y+6))\right] \; ;$$



Then,

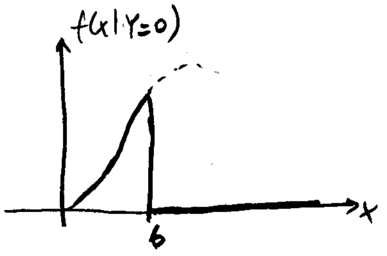
$$\int_{-\infty}^{\infty} f(y|x) f(x)\, dx = \int_{y-6}^{y+6} \left(\frac{1}{12}\right) f_X(x) dx = \int_{y-6}^{y+6} \frac{1}{12} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} dx$$

$$= \frac{1}{12}\left[-e^{-x^2/2\sigma^2}\right]_{y-6}^{y+6} = \frac{e^{-(y-6)^2/2\sigma^2} - e^{-(y+6)^2/2\sigma^2}}{12}$$

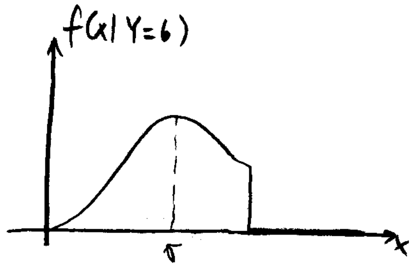
Hence, $f(x|y)$ is

$$f(x|y) = \begin{cases} \dfrac{f_X(x)}{e^{-(y-6)^2/2\sigma^2} - e^{-(y+6)^2/2\sigma^2}}, & y-6 \le x \le y+6 \\ 0, & \text{else} \end{cases}$$

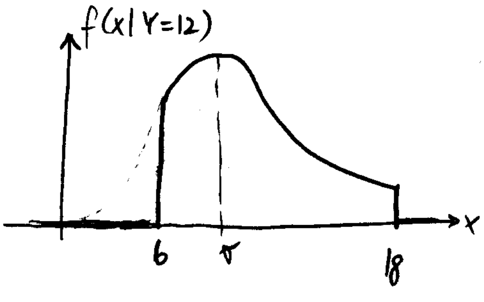
For $y=0$:



For $y=6$:



For $y=12$:



(b) For $y=12$, we must find the x at which $f(x|Y=12)$ is maximum:

$$\boxed{\hat{x} = \sigma = 7{,}979}$$

Problem 6.3 (3-2.4)

(a) We must find x s.t. $f(x|y)$ is maximum:

$$\frac{\partial}{\partial x}(f(x|y)) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{f(x,y)}{f(y)}\right) = 0 \quad \Rightarrow \frac{\partial}{\partial x}\left(f(x,y)\right) = 0 \quad \Rightarrow \frac{\partial}{\partial x}\left[K e^{-y^2} e^{-(x^2+4xy)}\right] = 0$$

$$\Rightarrow \cancel{K}(-2x+4y)\cancel{e^{-y^2}}\cancel{e^{-(x^2+4xy)}} = 0$$

$$\Rightarrow \boxed{\hat{x} = 2y}$$

(b) $\boxed{\hat{x} = 2.3 = 6}$

Problem 6.4 (3-3.2)

(a) From (3.40) in the textbook,

$$g(w,v) = |J| \underbrace{f(g^{-1}(w), h^{-1}(v))}_{\substack{f(g^{-1}(w))\, f(h^{-1}(v)) \\ \text{since } f(a,b) = f_X(a) f_Y(b)}} = \left| \begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial v} \end{vmatrix} \right| \cdot f_X(g^{-1}(w))\, f_Y(h^{-1}(v))$$

$$= \left| \begin{vmatrix} \frac{1}{g'(g^{-1}(w))} & 0 \\ 0 & \frac{1}{h'(h^{-1}(v))} \end{vmatrix} \right| \cdot f_X(g^{-1}(w))\, f_Y(h^{-1}(v))$$

$$= \frac{f_X(g^{-1}(w))}{|g'(g^{-1}(w))|} \; \frac{f_Y(h^{-1}(v))}{|h'(h^{-1}(v))|} = g_W(w)\, g_V(v), \text{ from (2.5) in textbook.}$$

Problem 6.5 (3-4.1)

(a) $\mu_X = \mu_Y = 0$, $\sigma_X^2 = 16$, $\sigma_Y^2 = 36$, $\rho_{XY} = 0{,}5$

$$Var[X+Y] = E\left[\left[(X+Y) - \overline{(X+Y)}\right]^2\right]$$

$$= E\left[\left[(X-\bar{X}) + (Y-\bar{Y})\right]^2\right]$$

$$= E[(X-\bar{X})^2] + E[(Y-\bar{Y})^2] + 2E[(X-\bar{X})(Y-\bar{Y})]$$

$$= \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y$$

$$= 16 + 36 + 2(0.5)(4)(6) = \boxed{76}$$

(b) $Var[X-Y] = E[(X-\bar{X})^2] + E[(Y-\bar{Y})^2] - 2E[(X-\bar{X})(Y-\bar{Y})]$

$$= \sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y = 16 + 36 - 2(0.5)(4)(6) = \boxed{28}$$

(c) $\sigma_{X+Y}^2 = 16 + 36 + 2(-0.5)(4)(6) = \boxed{28}$

$\sigma_{X-Y}^2 = 16 + 36 - 2(-0.5)(4)(6) = \boxed{76}$

Problem 6.6 (3-4.4)

(a) $\sigma_W^2 = E\left[\left[(X-\bar{X}) + (Y-\bar{Y}) + (Z-\bar{Z})\right]^2\right] = \sigma_X^2 + \sigma_Y^2 + \sigma_Z^2 + 2\rho_{XY}\sigma_X\sigma_Y + 2\rho_{XZ}\sigma_X\sigma_Z + 2\rho_{YZ}\sigma_Y\sigma_Z$

$$= 1 + 1 + 1 + 2(0)(1)(1) + 2(\tfrac{1}{2})(1)(1) + 2(\tfrac{-1}{2})(1)(1)$$

$$= \boxed{3}$$

(b) $$\rho_{WX} = \frac{E[WX]}{\sigma_W\sigma_X} = \frac{E[X^2] + E[XY] + E[XZ]}{\sigma_W\sigma_X} = \frac{1 + 0 + \frac{1}{2}}{\sqrt{3}\cdot 1} = \boxed{\frac{\sqrt{3}}{2} = 0{,}866}$$

(c) $$\rho_{W(Y+Z)} = \frac{E[W(Y+Z)]}{\sigma_W\sigma_{(Y+Z)}} = \frac{E[Y^2] + E[Z^2] + E[XY] + E[XZ] + 2E[YZ]}{\sqrt{3}\cdot\left(E[Y^2] + E[Z^2] + 2E[YZ]\right)^{1/2}} = \frac{1 + 1 + 0 + 1/2 + 2(-1/2)}{\sqrt{3}\cdot\left(1 + 1 + 2(-1/2)\right)^{1/2}}$$

$$= \frac{3/2}{\sqrt{3}} = \boxed{\frac{\sqrt{3}}{2} = 0{,}866}$$