Ilker R Capoglu

ECE 3075

HW #7

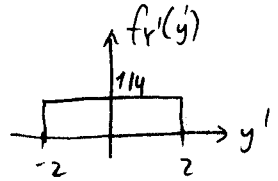
Solution

$\boxed{\text{Problem 1}}$ (3-5.1)

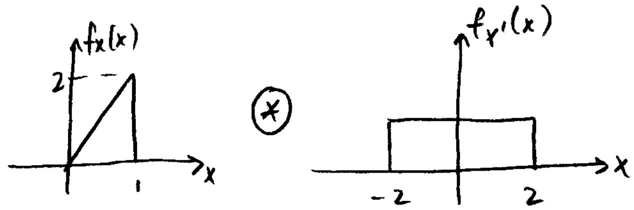
(a) $Z = X + 2Y = X + Y'$, where $Y' = 2Y$.

From simple transformation,

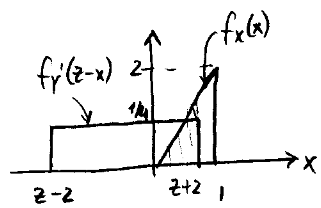
$f_{Y'}(y') = \text{uniform}(-2.0, 2.0)$:



$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_{Y'}(z-x)\,dx$$:

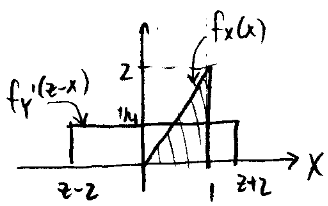


$-2 \le z \le -1$:



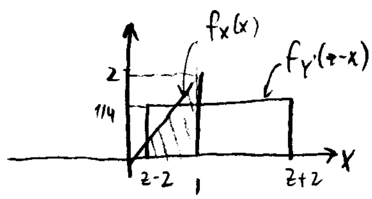
$$f_Z(z) = \int_0^{z+2} \frac{1}{4}(2x)\,dx = \frac{1}{4}(z+2)^2$$

$-1 \le z \le 2$:



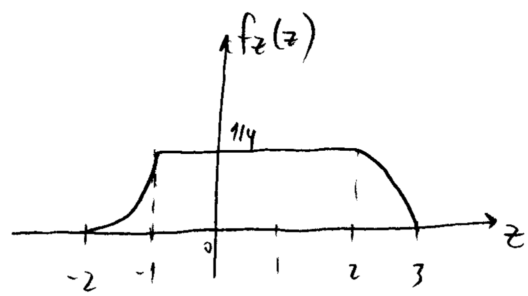
$$f_Z(z) = \int_0^1 \frac{1}{4}(2x)\,dx = \frac{1}{4}$$

$2 \le z \le 3$:



$$f_Z(z) = \int_{z-2}^{1} \frac{1}{4}(2x)\,dx = \frac{1}{4}(4z - z^2 - 3)$$

$$\Rightarrow f_Z(z) = \begin{cases} \frac{1}{4}(z+2)^2, & -2 \le z \le -1 \\ 1/4, & -1 \le z \le 2 \\ \frac{1}{4}(4z - z^2 - 3), & 2 \le z \le 3 \end{cases}$$



(b) $Pr\{0 \le z \le 1\} = \int_0^1 f_z(z)dz = 1 \times 1/4 = \boxed{1/4}$

$\boxed{\text{Problem 2}}$ (3-5.3)

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\Rightarrow \cos(100t+\theta)\cos(100t+\psi) = \underbrace{2\cos\left(\frac{\theta-\psi}{2}\right)}\cos\left(100t+\frac{\theta+\psi}{2}\right)$$

$$= A \quad \cos(100t+\phi)$$

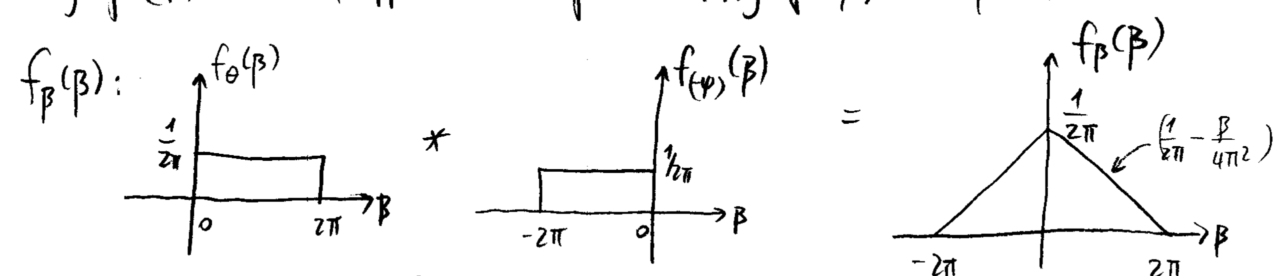
(a) let's call $\beta = \theta - \psi$. Then

$$Pr\{A>1\} = Pr\{2\cos\tfrac{\beta}{2} > 1\} = Pr\{|\tfrac{\beta}{2}| < \tfrac{\pi}{3}\} = Pr\{|\beta| < \tfrac{2\pi}{3}\}$$

Now let's find the density of β. Since $\beta = \theta + (-\psi)$ and the r.v.s θ and $(-\psi)$ are independent, we have

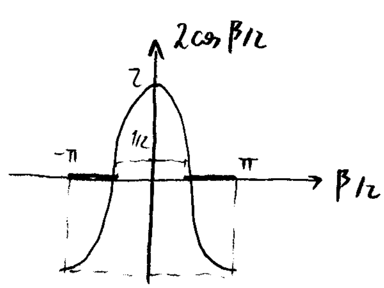
$$f_\beta(\beta) = (f_\theta * f_{-\psi})(\beta)$$

The density of $(-\psi)$ is the flipped version of the density of ψ. Therefore:



$$\Rightarrow Pr\{A>1\} = \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} f_\beta(\beta)d\beta = 2\int_0^{\frac{2\pi}{3}}\left(\frac{1}{2\pi} - \frac{\beta}{4\pi^2}\right)d\beta = \boxed{\frac{5}{9}}$$

(b) $Pr\{A \le \frac{1}{2}\} = Pr\{\cos\frac{\beta}{2} < \frac{1}{4}\} = Pr\{|\frac{\beta}{2}| > 1{,}318\} = Pr\{|\beta| > 2{,}636\}$

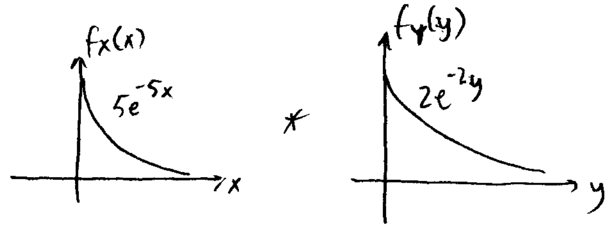


$$= 1 - \int_{-2{,}636}^{2{,}636} f_\beta(\beta)d\beta = 1 - 2\int_0^{2{,}636}\left(\frac{1}{2\pi} - \frac{\beta}{4\pi^2}\right)d\beta$$

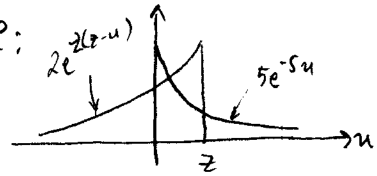
$$= \boxed{0{,}3369}$$

$\boxed{\text{Problem 3}}$ (3-5.5)

(a) $f_Z(z) = (f_X * f_Y)(z)$:



$\underline{z \geq 0}$:



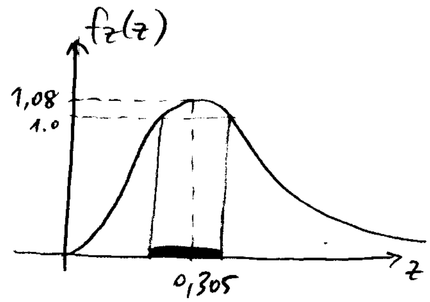
$$f_Z(z) = \int_0^z 5e^{-5u}\left(2e^{-2(z-u)}\right)du$$

$$= 10e^{-2z}\int_0^z e^{-3u}\,du = \frac{10}{3}e^{-2z}\left(1-e^{-3z}\right)$$

$$= \frac{10}{3}\left(e^{-2z}-e^{-5z}\right)$$

$$f_Z(0) = \frac{10}{3}\left(e^{-0}-e^{-0}\right) = \boxed{0}$$

(b) $f_Z(z) > 1$ for $\frac{10}{3}\left(e^{-2z}-e^{-5z}\right) > 1$;



(c) $$\Pr\{z > 0.1\} = \int_{0.1}^{\infty} \frac{10}{3}\left(e^{-2z}-e^{-5z}\right)dz = \frac{10}{3}\left[-\frac{1}{2}e^{-2z}\Big|_{0.1}^{\infty} + \frac{1}{5}e^{-5z}\Big|_{0.1}^{\infty}\right]$$

$$= \frac{10}{3}\left[-\frac{1}{2}\left(0-e^{-0.2}\right) + \frac{1}{5}\left(0-e^{-0.5}\right)\right]$$

$$= \boxed{0{,}9602}$$

$\boxed{\text{Problem 4}}$ (3-6.1)

Given $Z = X+Y$
Let $W = X$
} let's find $f_{ZW}(z,w)$ first.

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 \Rightarrow |J| = 1.$$

$X = \psi_1(z,w) = W$

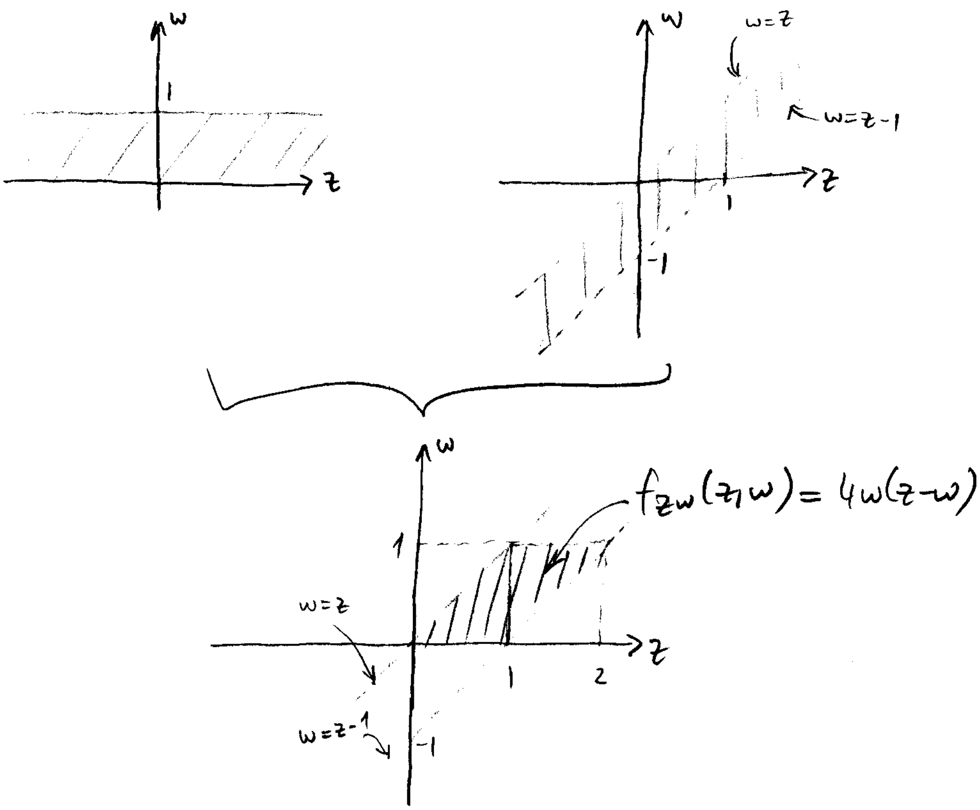
$Y = \psi_2(z,w) = z - W$

$$f_{ZW}(z,w) = |J|\, f_{XY}(\psi_1(z,w), \psi_2(z,w)) = f_{XY}(w, z-w) = 4w(z-w)$$

On which region of the z-w plane is $f_{ZW}(z,w)$ defined? This depends on what the region of $f_{XY}(x,y)$ is on the x-y plane.

$0 < x < 1 \Rightarrow 0 < w < 1$ (first limitation)

$0 < y < 1 \Rightarrow 0 < z-w < 1 \Rightarrow z-1 < w < z$ (second limitation)



$$f_Z(z) = \int_{w=-\infty}^{\infty} f_{ZW}(z,w)\,dw = \begin{cases} \displaystyle\int_0^z 4w(z-w)\,dw, & 0<z<1 \\ \displaystyle\int_{z-1}^1 4w(z-w)\,dw, & 1<z<2 \end{cases}$$

$$= \boxed{\begin{cases} 2z^3/3, & 0<z<1 \\ \frac{2}{3}[-z^3+6z-4], & 1<z<2 \end{cases}}$$

$\boxed{\text{Problem 5}}$ (3-7.4)

(a) $\phi_X(u) = \frac{5}{5-ju}$, $\phi_Y(u) = \frac{2}{2-ju}$

$$\phi_Z(\omega) = \frac{5}{5-ju} \cdot \frac{2}{2-ju} = \frac{-10/3}{5-ju} + \frac{10/3}{2-ju}$$

$$\Rightarrow \boxed{f_Z(z) = \frac{10}{3}\left[e^{-2z} - e^{-5z}\right]u(z)}$$

(b) $$E[Z] = \frac{1}{j}\frac{d\phi_Z(u)}{du}\Big|_{u=0} = \frac{1}{j}\left[\frac{d}{du}\frac{10}{10-j7u-u^2}\right]_{u=0} = \frac{0-10(-j7-2u)}{j(10-j7u-u^2)^2}\Big|_{u=0}$$

$$= \boxed{0.7}$$

$$E[Z^2] = \frac{1}{j^2}\left[\frac{d^2}{du^2}\phi_Z(u)\right]_{u=0} = \frac{1}{j^2}\left[\frac{d}{du}\frac{j70+20u}{(10-j7u-u^2)^2}\right]_{u=0}$$

$$= \frac{1}{j^2}\,\frac{20(10-j7u-u^2)^2 - (j70+20u)(2)(10-j7u-u^2)(-j7-2u)}{(10-j7u-u^2)^4}\Bigg|_{u=0}$$

$$= \frac{1}{-1}\,\frac{20(10)^2 - (j70)(2)(10)(-j7)}{10^4} = \boxed{0{,}78}$$