ECE 8873

Data Compression and Modeling

Lecture 9: Coding of Model Parameters

School of Electrical and Computer Engineering
Georgia Institute of Technology
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Linear Prediction

$e_n = x_n - \tilde{x}_n = x_n - \sum_{i=1}^{p} a_i x_{n-i}$ is the prediction error.

Choose mean squared error as the performance measure

$$\mathrm{E} = E[(X_n - \tilde{X}_n)^2] = E\left[\left(X_n - \sum_{i=1}^{p} a_i X_{n-i}\right)^2\right]$$

$$\frac{\partial}{\partial a_i}\mathrm{E} = \frac{\partial}{\partial a_i} E[(X_n - \tilde{X}_n)^2] = E\left[2(X_n - \tilde{X}_n)\frac{\partial[-\tilde{X}_n]}{\partial a_i}\right] = 0$$

$$\Rightarrow E[(X_n - \tilde{X}_n)X_{n-i}] = 0$$

$$\begin{bmatrix} r(1) \\ r(2) \\ r(3) \\ \vdots \\ r(p) \end{bmatrix} = \begin{bmatrix} r(0) & r(1) & r(2) & \cdots & r(p-1) \\ r(1) & r(0) & r(1) & \cdots & r(p-2) \\ r(2) & r(1) & r(0) & \cdots & r(p-3) \\ \cdots & & \cdots & & \cdots \\ r(p-1) & r(p-2) & r(p-3) & & r(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_p \end{bmatrix}$$

The set of normal equations, Yule-Walker equations, or Wiener-Hopf equations

Solving the Normal Equations

$\mathrm{E}^{(0)} = r(0)$ Then for $i = 1, 2, \cdots, p$ iterate:

$$k_i = \left\{ r(i) - \sum_{j=1}^{i-1} a_j^{(i-1)} r(|i-j|) \right\} \left[\mathrm{E}^{(i-1)} \right]^{-1}$$

$$a_i^{(i)} = k_i \quad \text{and} \quad a_j^{(i)} = a_j^{(i-1)} - k_i a_{i-j}^{(i-1)}, \qquad j = 1, 2, \cdots, i-1$$

$$\mathrm{E}^{(i)} = (1 - k_i^2)\mathrm{E}^{(i-1)}$$

Show example for $p = 2$

- Efficient algorithms exist for solving the equations.
- k is called the reflection coefficient and is better than the predictor coefficients for coding purposes; value bounded in (-1,1) for stable all-pole filter.

All-Pole Model Parameters

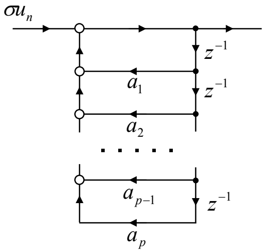
$$X(\omega) \Rightarrow \frac{\sigma^2}{|A(z)|^2}$$

- Predictor coefficients $\{a_i\}_{i=1}^{p}$ Stability?
- Reflection coefficients $\{k_i\}_{i=1}^{p}$ $|k_i| < 1$
- Log area ratio $\left\{\log \frac{1-k_i}{1+k_i}\right\}_{i=1}^{p}$
- Autocorrelation $\{r(i)\}_{i=0}^{p}$ or $\left\{\frac{r(i)}{r(0)}\right\}_{i=1}^{p}$
- Cepstral coefficients $\{c_i\}_{i=1}^{p}$
- Line spectral frequecies $\{f_i\}_{i=1}^{p}$

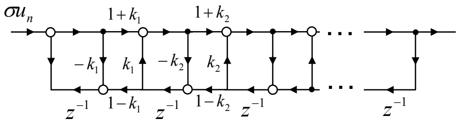
Model Parameters & System Structures

all-pole model: $\dfrac{\sigma}{A(z)}$

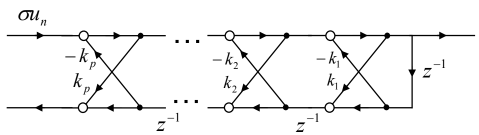
$$x_n = \sum_{i=1}^{p} a_i x_{n-i} + \sigma u_n$$ where u_n is a unit variance innovation sequence



Direct form



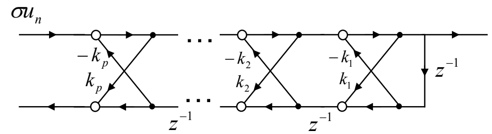
Kelly-Lochbaum model



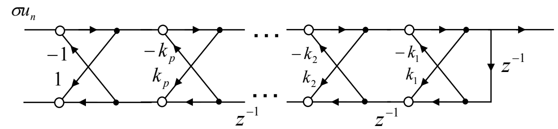
Lattice form

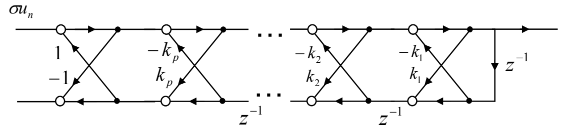
Extreme Boundary Conditions

Lattice form



What'll happen if -

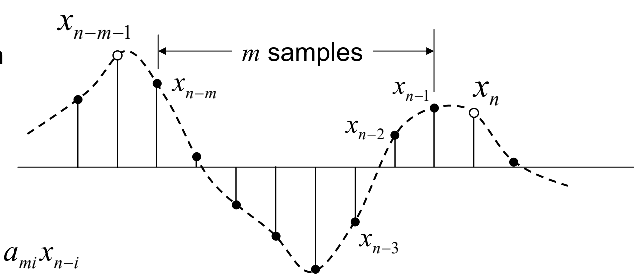




Forward & Backward Prediction

- (Do not confuse this with forward and backward **adaptation**.)

replace p with m and M



$$e_n^+ = x_n + \sum_{i=1}^{m} a_{mi} x_{n-i}$$

$$e_n^- = x_{n-m-1} + \sum_{i=1}^{m} b_{mi} x_{n-i}$$

$$E^+(z) = X(z)A_m(z) = X(z)\left[\sum_{i=0}^{m} a_{mi} z^{-i}\right]$$

$$E^-(z) = X(z)B_m(z) = X(z)\left[\sum_{i=1}^{m+1} b_{mi} z^{-i}\right]$$

Line Spectral Pairs (LSP)

Forward & back recursion in Linear Prediction

For $m = 1, 2, \cdots, M$

$$A_m(z) = A_{m-1}(z) + k_m B_{m-1}(z) \qquad B_m(z) = z^{-(m+1)} A_m(z^{-1})$$

$$zB_m(z) = k_m A_{m-1}(z) + B_{m-1}(z) \qquad A_0(z) = 1, \quad zB_0(z) = 1$$

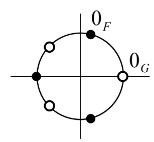
Let $k_{M+1} = \pm 1$, which correspond to completely closed and completely open boundary conditions, respectively, and create two polynomials:

$k_{M+1} = 1$ ⇒ $F(z) = A_{M+1}(z) = A_M(z) + B_M(z) = A_M(z) + z^{-(M+1)} A_M(z^{-1})$

$k_{M+1} = -1$ ⇒ $G(z) = A_{M+1}(z) = A_M(z) - B_M(z) = A_M(z) - z^{-(M+1)} A_M(z^{-1})$

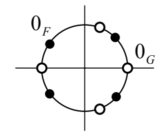
M even:

$G(1) = A_M(1) - B_M(1)$
$= A_M(1) - A_M(1) = 0$
$F(-1) = A_M(-1) - A_M(-1)$
$= 0$



M odd:

$G(1) = A_M(1) - B_M(1)$
$= A_M(1) - A_M(1) = 0$
$G(-1) = A_M(-1) - A_M(-1)$
$= 0$



Line Spectral Polynomials

$$F(z) = A_M(z) + B_M(z) = A_M(z) + z^{-(M+1)} A_M(z^{-1})$$
$$G(z) = A_M(z) + B_M(z) = A_M(z) - z^{-(M+1)} A_M(z^{-1})$$

$$A_M(z) = \frac{1}{2}\left[F(z) + G(z)\right]$$

$F(z)$ and $G(z)$ can have roots only on the unit circle.

Proof: $F(z) = A_M(z) + z^{-(M+1)} A_M(z^{-1})$

If z_0 satisfies $F(z_0) = 0$, $\quad F(z_0) = A_M(z_0) + z_0^{-(M+1)} A_M(z_0^{-1})$

$$\Rightarrow \quad z_0 \frac{z_0^M A_M(z_0)}{A_M(z_0^{-1})} = -1 \qquad \left(=1 \text{ for the case of } G(z) = 0\right)$$

Since $1/A_m(z)$ are all stable all-pole filters; $A_m(z)$ has roots within the unit circle.

$$H_m(z) = z^m \prod_{i=1}^{m} \frac{(1 - z_{mi} z^{-1})}{(1 - z_{mi}^* z)} = \prod_{i=1}^{m} \frac{(z - z_{mi})}{(1 - z_{mi}^* z)}, \quad |H_m(z)|^2 = \prod_{i=1}^{m} \left|\frac{(z - z_{mi})}{(1 - z_{mi}^* z)}\right|^2$$

$$\left|\frac{z - z_{mi}}{1 - z_{mi}^* z}\right|^2 = \frac{zz^* - z_{mi} z^* - z_{mi}^* z + z_{mi} z_{mi}^*}{1 - z_{mi} z^* - z_{mi}^* z + z_{mi} z_{mi}^* z z^*}, \quad |z - z_{mi}|^2 - |1 - z_{mi}^* z|^2 = (|z|^2 - 1)(1 - |z_{mi}|^2)$$

Line Spectral Polynomials

Proof cont'd

$$\left|\frac{z - z_{mi}}{1 - z_{mi}^* z}\right|^2 = \frac{zz^* - z_{mi} z^* - z_{mi}^* z + z_{mi} z_{mi}^*}{1 - z_{mi} z^* - z_{mi}^* z + z_{mi} z_{mi}^* z z^*}, \quad |z - z_{mi}|^2 - |1 - z_{mi}^* z|^2 = (|z|^2 - 1)(1 - |z_{mi}|^2)$$

Since $|z_{mi}| < 1$ for $1/A_m(z)$ to be stable; $|H_m(z)|$ is thus

$$|H_m(z)| : \begin{cases} > 1 & \text{if } |z| > 1 \\ < 1 & \text{if } |z| < 1 \\ = 1 & \text{if } |z| = 1 \end{cases} \qquad \text{for } m = 1, 2, \cdots, M$$

If z_0 satisfies $F(z_0) = 0$ or $G(z_0) = 0 \quad \Rightarrow \quad \left| z_0 \frac{z_0^M A_M(z_0)}{A_M(z_0^{-1})} \right| = |z_0| \, |H_M(z_0)| = 1$

Therefore $|z_0| = 1 \quad \Rightarrow$ Roots are on the unit circle.

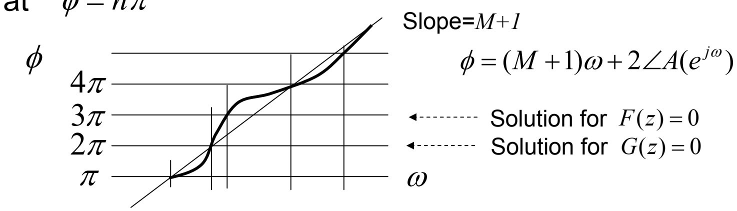
Other Properties

$$F(z) = A_M(z) + z^{-(M+1)} A_M(z^{-1}) = \prod_{i=1}^{M+1} (1 - z_{fi} z^{-1}) \quad \Rightarrow \quad \prod_{i=1}^{M+1} (-z_{fi}) = 1 = a_0$$
$$G(z) = A_M(z) - z^{-(M+1)} A_M(z^{-1}) = \prod_{i=1}^{M+1} (1 - z_{gi} z^{-1}) \quad \Rightarrow \quad \prod_{i=1}^{M+1} (-z_{gi}) = -1 = -a_0$$

$$F(z) = A_M(z) + z^{-(M+1)} A_M(z^{-1})$$
$$G(z) = A_M(z) - z^{-(M+1)} A_M(z^{-1})$$

What is $\dfrac{e^{j(M+1)\omega} A_M(e^{j\omega})}{A_M(e^{-j\omega})} = e^{j\phi}$?

Solutions for $F(z) = 0$ and $G(z) = 0$ occurs alternately at $\phi = n\pi$



The ϕ Function

$$\frac{e^{j(M+1)\omega} A_M(e^{j\omega})}{A_M(e^{-j\omega})} = e^{j\phi}$$

$$A_M(e^{j\omega}) = \prod_{i=1}^{M} (1 - z_i e^{-j\omega}) \qquad \angle A_M(e^{j\omega}) = \sum_{i=1}^{M} \angle(1 - z_i e^{-j\omega})$$

$$z_i = r_i e^{j\theta_i} \qquad 0 < r_i < 1$$

$$1 - z_i e^{-j\omega} = 1 - r_i \cos(\theta_i - \omega) - j r_i \sin(\theta_i - \omega)$$

$$\text{Re}\{1 - z_i e^{-j\omega}\} = 1 - r_i \cos(\theta_i - \omega) > 0 \qquad \text{Im}\{1 - z_i e^{-j\omega}\} = -r_i \sin(\theta_i - \omega)$$

Let $\angle(1 - z_i e^{-j\omega}) = \rho_i$. ρ_i is always in I or IV quadrand.

$$\rho_i = \tan^{-1} \frac{-r_i \sin(\theta_i - \omega)}{1 - r_i \cos(\theta_i - \omega)}$$

The Phase Function of Individual Poles

$$\frac{d\rho_i}{d\omega} = \frac{r_i \cos(\theta_i - \omega) - r_i^2}{1 - 2r_i \cos(\theta_i - \omega) + r_i^2} \qquad \frac{d\rho_i}{d\omega} = 0 \ \text{ when } \cos(\theta_i - \omega) = r_i \ \text{ or } \ \omega = \theta_i \pm \cos^{-1} r_i$$

$$\cos(\theta_i - \omega) = r_i \ \text{ and } \ \sin(\theta_i - \omega) = \pm\sqrt{1 - r_i^2}$$

Thus,

$$(\rho_i)_{extremum} = \tan^{-1} \frac{-r_i \pm \sqrt{1 - r_i^2}}{1 - r_i^2} = \tan^{-1} \frac{\pm r_i}{\sqrt{1 - r_i^2}},$$

$$(\rho_i)_{\max} = \tan^{-1} \frac{r_i}{\sqrt{1 - r_i^2}}, \quad (\rho_i)_{\min} = -\tan^{-1} \frac{r_i}{\sqrt{1 - r_i^2}}$$

$$\left(\frac{d\rho_i}{d\omega}\right)_{\min} = \frac{-r_i - r_i^2}{1 + 2r_i + r_i^2} = \frac{-r_i}{1 + r_i},$$

$$\left(\frac{d\rho_i}{d\omega}\right)_{\max} = \frac{r_i - r_i^2}{1 + 2r_i + r_i^2} = \frac{r_i}{1 - r_i}$$

When a pole is close to the unit circle, the phase function ρ_i has a large slope and will cross $n\pi$ lines more rapidly, resulting in denser LSPs.

Overall Phase Function

$$\phi = (M+1)\omega + 2\angle A(e^{j\omega}) = (M+1)\omega + 2\sum_{i=1}^{M} \rho_i$$

$$\frac{d\phi}{d\omega} = (M+1) + 2\sum_{i=1}^{M} \frac{d\rho_i}{d\omega} \qquad \left(\frac{d\rho_i}{d\omega}\right)_{\min} = \frac{-r_i}{1 + r_i}$$

$$\frac{d\phi}{d\omega} = (M+1) + 2\sum_{i=1}^{M} \frac{d\rho_i}{d\omega} \geq 1 + \sum_{i=1}^{M} \left[1 + 2\left(\frac{d\rho_i}{d\omega}\right)_{\min}\right] = 1 + \sum_{i=1}^{M} \left[1 - \frac{2r_i}{1 + r_i}\right] \geq 1$$

$$\phi(\omega)\big|_{\omega=0} = 0 \quad \text{and} \quad \phi(\omega)\big|_{\omega=2\pi} = (M+1) \bullet 2\pi$$

ϕ is a monotonic increasing function of ω, intersecting $n\pi$ sequentially and non-repetitively. Therefore, solutions to $F(z) = 0$ and $G(z) = 0$ occur interleavingly.

Recap

- LSPs correspond to the roots of the two polynomials associated with extreme boundary conditions in all-pole systems.
- All roots of the two polynomials are on the unit circle and if we scan the unit circle from 0 to 2π , we encounter the roots of the polynomials interleavingly.
- LSPs are good candidate for transmission to provide side information about the predictor.