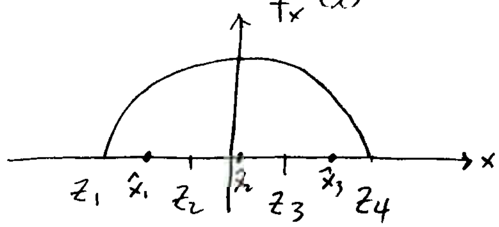
Final Takehome Exam. Soo Hyun Bae.

Problem 1.

(a).



The minimum MSE of this quantizer is

$$D = E\left[\,|X - Q(x)|^2\right]$$

$$= \sum_i \int_{R_i} (x-\hat{x}_i)^2 f_X(x)\,dx \qquad \text{when } \hat{x}_i = \frac{\int_{R_i} x f_X(x)\,dx}{\int_{R_i} f_X(x)\,dx} \qquad -①$$

Since the pdf is symmetric, it is true that

$$\begin{cases} z_2 = -z_3 \\ \hat{x}_1 = -\hat{x}_3 \\ \hat{x}_2 = 0 . \end{cases}$$

Hence,

$$D = 2\int_{-\pi/2}^{z_2} (x-\hat{x}_1)^2 f_X(x)\,dx + 2\int_{z_2}^{0} x^2 f_X(x)\,dx$$

$$= 2\int_{-\pi/2}^{z_2} (x^2 - 2\hat{x}_1 x + \hat{x}_1^2)\, f_X(x)\,dx + 2\int_{z_2}^{0} x^2 f_X(x)\,dx$$

$$= 2\int_{-\pi/2}^{0} x^2 f_X(x)\,dx - 2\int_{-\pi/2}^{z_2} (-2\hat{x}_1 x + \hat{x}_1^2)\, f_X(x)\,dx \qquad -②$$

b) From ⊖ and the result, the minimum MSE is

$$D = 2\int_{-\pi/2}^{0} x^2 f(x)\,dx - 2\int_{-\pi/2}^{-0.4065} (-2\times(0.8130)x + (-0.8130)^2 f_X(x)\,dx$$

$$= 0.0426$$

c) $$p(\hat{x}_1) = \int_{-\pi/2}^{-0.4065} f_X(x)\,dx = 0.3023$$

$$p(\hat{x}_2) = \int_{-0.4065}^{0.4065} f_X(x)\,dx = 0.3954$$

$$p(\hat{x}_3) = p(\hat{x}_1) = 0.3023.$$

d) $$E[\hat{x}] = \hat{x}_1 p(\hat{x}_1) + \hat{x}_2 p(\hat{x}_2) + \hat{x}_3 p(\hat{x}_3) = 0$$

e) $$E[\hat{x}^2] = \hat{x}_1^2 p(\hat{x}_1) + \hat{x}_2 p(\hat{x}_2) + \hat{x}_3 p(\hat{x}_3) = 0.3986$$

f). When the probability of $\hat{x}_i$ are uniformly distributed, the entropy is maximized.

So we want to find the z_2 which satisfies

$$\int_{-\pi/2}^{z_2} f_X(x)\,dx = \int_{z_2}^{-z_2} f_X(x)\,dx$$

$$\boxed{\text{cf. } \int_a^b \tfrac{1}{2}\cos x\,dx = \tfrac{1}{2}(\sin b - \sin a)}$$

$$\Rightarrow \tfrac{1}{2}(\sin z_2 - \sin(-\pi/2)) = \tfrac{1}{2}(\sin(-z_2) - \sin z_2)$$

$$\Rightarrow \sin z_2 + 1 = -2\sin z_2$$

$$\therefore \sin z_2 = \tfrac{1}{3} \qquad \therefore z_2 = -0.3398$$

$$\therefore \{z_1, z_2, z_3, z_4\} = \{-\pi/2, -0.3398, 0.3398, \pi/2\}$$

$$\{\hat{x}_1, \hat{x}_2, \hat{x}_3\} = \{-0.7720, 0, 0.7720\}$$

From ① $\hat{x}_1$ is function of z_2, let's write it $\alpha(z_2)$.

to find z_2 which minimizes D, take derivative and make it to zero. w.r.t z_2

From ②, $$D = 2\int_{-\pi/2}^{0} x^2 f_x(x)dx + 4\alpha(z_2)\int_{-\pi/2}^{z_2} x f_x(x)dx - 2\alpha^2(z_2)\int_{-\pi/2}^{z_2} f_x(x)dx.$$

$$\frac{\partial D}{\partial z_2} = 4\frac{\partial \alpha(z_2)}{\partial z_2}\int_{-\pi/2}^{z_2} x f_x(x)dx + 4\alpha(z_2)\, z_2 \tfrac{1}{2}\cos z_2$$

$$- 4\alpha(z_2)\frac{\partial \alpha(z_2)}{\partial z_2}\int_{-\pi/2}^{z_2} f_x(x)dx - 2\alpha^2(z_2)\tfrac{1}{2}\cos z_2 = 0.$$

$$= 4\frac{\partial \alpha(z_2)}{\partial z_2} \times \left\{\tfrac{1}{2} z_2 \sin z_2 - \tfrac{\pi}{4} + \tfrac{1}{2}\cos z_2\right\} + 4\alpha(z_2) z_2 \tfrac{1}{2}\cos z_2$$

$$- 4\alpha(z_2)\frac{\partial (z_2)}{\partial z_2} \times \tfrac{1}{2}(\sin z_2 + 1) - 2\alpha^2(z_2)\tfrac{1}{2}\cos z_2$$

when $$\alpha(z_2) = \frac{\int_{-\pi/2}^{z_2} x f_x(x)dx}{\int_{-\pi/2}^{z_2} f_x(x)dx} = \frac{\tfrac{1}{2} z_2 \sin z_2 - \tfrac{\pi}{4} + \tfrac{1}{2}\cos z_2}{\tfrac{1}{2}(\sin z_2 + 1)}$$

Using matlab $z_2 = -0.4065$ (Please see figure 1).

And it is verified in figure 2 and figure 3. This implementation uses Lloyd iteration

$\{z_1, z_2, z_3, z_4\} = \{-\pi/2, -0.4065, 0.4065, \pi/2\}$

$\{x_1, x_2, x_3\} = \{-0.8130, 0, 0.8130\}$

```
>> syms z alpha dalpha dd;
alpha = (1/2*z*sin(z)+1/2*cos(z)-pi/4)/(1/2*sin(z)+1/2);
dalpha = diff(alpha,z);
dd    =    4*dalpha    *    (0.5*z*sin(z)-pi/4+0.5*cos(z))    +    4*alpha*z*0.5*cos(z)    -
4*alpha*dalpha*0.5*(sin(z)+1) - 2*alpha^2*0.5*cos(z);
solve(dd,z)


ans =


[                                   1/2*pi]
[ sin(-.41867779090385401205172182369 48)
```

Figure 1. A MATLAB *program for find z2 which minimizes the MSE in quantization*

```
NoOfIteration = 200;levels = 3;StartStepSize = 1;
xpx = inline('x .* 0.5 .* cos(x)');px = inline('0.5 .* cos(x)');
t = zeros(1, levels+1);yi = zeros(1, levels);
% Set the initial levels
t(1) = -pi/2;t(levels + 1) = pi/2;
for i=2:levels
    t(i) = (i-2) * StartStepSize - (levels-1)*(StartStepSize)/2 + StartStepSize/2;
end
disp(sprintf('\nInitial levels : '));disp(t);

for i=1:NoOfIteration
    for j=1:levels-1
        y_i = quadl(xpx, t(j), t(j+1)) / quadl(px, t(j), t(j+1));
        y_in = quadl(xpx, t(j+1), t(j+2)) / quadl(px, t(j+1), t(j+2));
        x_i = (y_i + y_in)/2;
        t(j+1) = x_i;
    end
end
disp(sprintf('\nFinal levels : '));disp(t);

for i=1:levels
     y(i) = quadl(xpx, t(i), t(i+1)) / quadl(px, t(i), t(i+1));
end
disp(sprintf('\nReproduction values : '));disp(y);

MSE = 0;
for i=1:levels-1
     y_i = quadl(xpx, t(j), t(j+1)) / quadl(px, t(j), t(j+1));
     vpx = inline(sprintf('(x - %d).^2 .* 0.5 .* cos(x)', y_i));
     MSE = MSE + quadl(vpx, t(j), t(j+1));
end

disp(sprintf('MSE = %d', MSE));
```

Figure 2. A MATLAB *program for Lloyd iteration to design optimal quantizer*

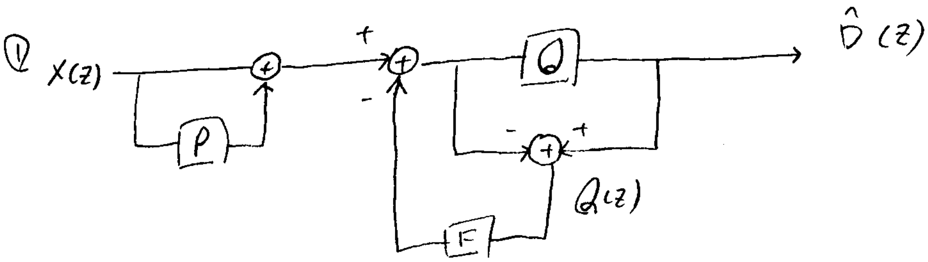
```
Initial levels :
   -1.5708    -0.5000     0.5000     1.5708

Number of iteration : 200

Final levels :
   -1.5708    -0.4065     0.4065     1.5708

Reproduction values :
   -0.8130     0.0000     0.8130

MSE = 4.258789e-002
```

Figure 3. Optimum quantizer and reproduction values

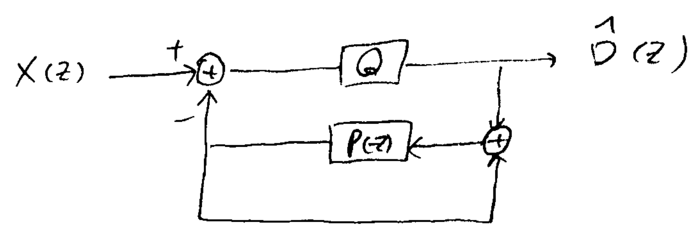
Problem 2. -2

a). Let's see the encoder part first.

i) Encoder.

① 

$$\hat{D}(z) = X(z)(1-P(z)) + Q(z)(1-F(z))$$

② 

$$\hat{D}(z) = X(z)\frac{1}{1+\frac{P(z)}{1-P(z)}} + Q(z)\frac{1}{1+\frac{P(z)}{1-P(z)}}$$

$$= X(z)(1-P(z)) + Q(z)(1-P(z))$$

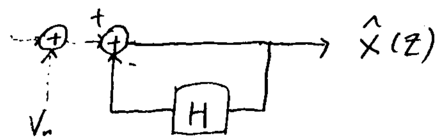
∴ when $F(z) = P(z)$, these two schemes are equivalent.

ii). Decoder.

When $P(z) = H(z)$, the two decoder schemes are equivalent.

Therefore, when $F(z) = P(z) = H(z)$, the two encoder/decoder schemes are equivalent.

b). In decoder part



If there is no channel noise

$$\hat{x}(z) = \left\{ x(z)(1-p(z)) + Q(z)(1-F(z)) \right\} \times \frac{1}{1-H(z)}$$

when $H(z) = p(z)$

$$\hat{x}(z) = x(z) + Q(z)\frac{1-F(z)}{1-H(z)} \quad \text{—— ①}$$

If there is the channel noise

$$\hat{x}(z) = x(z)\frac{1-P(z)}{1-H(z)} + Q(z)\frac{1-F(z)}{1-H(z)} + V(z)\frac{1}{1-H(z)} \quad \text{— ②}$$

when ① has the "optimum" noise shaping,

$F = F_o$
$H = H_o$ for ①, ②.

Assume that the quantization noise and the channel noise are white. The SQNRs are

$$SQNR_{WOH} = \frac{\int_{all} S_x(f)\, df}{\int_{all} \sigma_q^2 \left| \frac{1-F_o(z)}{1-H_o(z)} \right|^2 df}$$

without noise

$$SQNR_{CH} = \frac{\int_{all} S_x(f) \left| \frac{1-P(z)}{1-H_o(z)} \right|^2 df}{\int_{all} \sigma_q^2 \left| \frac{1-F_o(z)}{1-H_o(z)} \right|^2 + \sigma_v^2 \left| \frac{1}{1-H_o(z)} \right|^2 df}$$

channel noise

In terms of SQNR, we do not have any control for both noises, but we can adjust the signal power.

Let's define the inverse of SQNRs.

$$\frac{1}{SQNR_{WoN}} \triangleq C_{WoN} = \frac{\sigma_q^2 \left|\frac{1-F_0(z)}{1-H_0(z)}\right|^2}{S_x(f)}$$

$$\frac{1}{SQNR_{CN}} \triangleq C_{CN} = \frac{\sigma_q^2 \left|\frac{1-F_0(z)}{1-H_0(z)}\right|^2}{S_x(f) \left|\frac{1-P(z)}{1-H_0(z)}\right|^2} + \frac{\sigma_v^2 \left|\frac{1}{1-H_0(z)}\right|^2}{S_x(f) \left|\frac{1-P(z)}{1-H_0(z)}\right|^2}$$

$$= \frac{\sigma_q^2 \left|\frac{1-F_0(z)}{1-H_0(z)}\right|^2}{S_x(f)} \times \left|\frac{1-H_0(z)}{1-P(z)}\right|^2 + \quad ''$$

$$= C_{WoN} \times \left|\frac{1-H_0(z)}{1-P(z)}\right|^2 + \quad ''$$

To keep $C_{CN} > C_{WoN}$,

$$C_{WoN} \times \frac{|1-H_0(z)|^2}{|1-P(z)|^2} + \frac{\sigma_v^2 \left|\frac{1}{1-H_0(z)}\right|^2}{S_x(f) \left|\frac{1-P(z)}{1-H_0(z)}\right|^2} > C_{WoN}$$

$$\therefore \left\{ C_{WoN} \times |1-H_0(z)|^2 + \frac{\sigma_v^2}{S_x(f)} \right\} \times \frac{1}{C_{WoN}} > |1-P(z)|^2.$$

$$\Rightarrow |1-H_0(z)|^2 + \frac{\sigma_v^2}{S_x(f)} \frac{S_x(f)\,|1-H_0(z)|^2}{\sigma_q^2\,|1-F_0(z)|^2}$$

$$= |1-H_0(z)|^2 \times \left\{ 1 + \frac{\sigma_v^2}{\sigma_q^2} \frac{1}{|1-F_0(z)|^2} \right\} > |1-P(z)|^2 \qquad \text{— ③}$$

Therefore, we can design $P(z)$ to maintain or get better SQNR in the presence of channel noise.

In terms of the noise shaping, we have the ill-shaped additive noise from $F_0(z)$, $H_0(z)$. Therefore, to deal with the channel noise in terms of noise shaping, we should redesign the $F(z)$ & $H(z)$ again.

yes, but how?

Problem 3.

i) When and where was the technique first proposed in the literature?

The theoretical development of optimal vector quantization was proposed in "Asymptotically Optimal Block Quatization" by A. Gersho in IEEE trans. Information Theory, 1979.

Based on this backround, "Multiple Stage Vector Quatization for Speech Coding" was published by B.W. Juang in ICASSP 1982.

ii).

① G.722 , G.722.2 Recommendation (01/2002)

Since this speech coder supports various modes of bandwidth the multi-stage vector quantizer is combined with split vector quantization

The residual vector $\underline{r}(n)$ is

$$\underline{r}(n) = \underline{z}(n) - \underline{p}(n)$$

where $\underline{z}(n)$ is the mean-removed ISF vector

$\underline{p}(n)$ is the predicted LSF vector

The predicted LSF vector is defined as

$$\underline{p}(n) = \frac{1}{3}\hat{\underline{r}}(n-1)$$

In here, the ISF residual vector $\underline{r}$ is quantized using split-multistage vector quantization which splits the vector into 2 subvectors, $\underline{r}_1(n)$ and $\underline{r}_2(n)$ of dimensions 9 and 7. These two vectors are quantized in two stages. First they are quantized and the quantization error vectors $\underline{r}^{(2)} = \underline{r} - \hat{\underline{r}}_i$ are split in the next stage into 3 and 2 vectors, and then they are quantized using the different bits with correspond to the bit rates.

②. G.723.1 (ITU-T Recommendation Draft of 1995-OCT)

The quantization stages in G.723.1 are divided into two parts, one is the quantization of the residual signal and the other is of the parameters estimation as follow

$$e(n) = \underline{r}(n) - \underline{r}'(n) = r(n) - G\sum_{k=0}^{M-1} g_k h[n-m_k]$$

After the first quantization of $\underline{r}(n)$, the estimated gain G is quantized by a logarithmic quantizer with 24 steps.

③. G.729 ITU-T Recommendation (03/96)

The quantization coefficients are obtained from the sum of two codebooks

$$\hat{l}_i = \begin{cases} L1_i(L1) + L2_i(L2) & i = 1 \cdots 5 \\ L1_i(L1) + L3_{i-5}(L3) & i = 6 \cdots 10 \end{cases}$$

where L1, L2, L3 are the codebook indices.

The first stage is a 10-dimensional VQ using codebook L1 with 7 bits. The second stage is a 10 bit VQ which has been implemented as a split VQ using two 5-dimensional codebooks, L2 and L3 containing 32 entries (5 bits) each.

④ JPEG 2000, "JPEG2000: Standard for Interactive Image" by David S. Taubman et la.

In order to achieve the desired code length, JPEG2000 adjust the quantization parameters in code blocks. The quantizer for the block is

$$q_b^{[p]}[j] = \mathrm{sign}(y_b[j]) \left\lfloor \frac{|y_b[j]|}{2^p \Delta_b} \right\rfloor$$

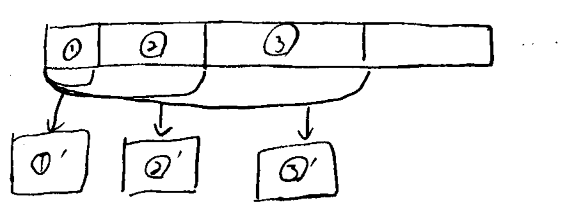
The rate-distortion optimizer would reduce some bits according to the desired rate.

Problem 4.

The scalability refers to the capability of recovering physically meaningful image or video information by decoding only partial compressed bit streams. The term "scalability" not only stands for the "bandwidth scalability" but also channel error characteristics scalability and computing power scalability.

Video signal can be made scalable in terms of following four views, quality scalability. spatial scalability. temporal scalability and frequency scalability

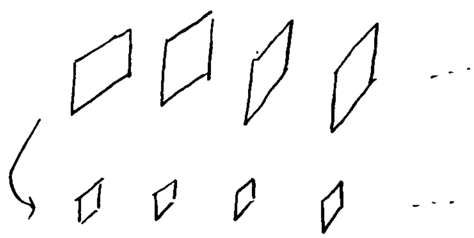
* In quality scalability, Video signal is represented with varying accuracies in color patterns Please see the following figure.



In encoder, the video frame is DCT-ed and quantized at base level①. Then, the quantization error is quantized again to obtain the bit streams, ② & ③ & ...

As the decoder processes more bit streams, the quality of the video signal is getting better

* In spatial Scalability, video signal is defined as the representation of the same video in varying spatial resolutions or sizes.



Encoder comes up with bit streams for various resolution layers, so the decoder works on the first layer, the user can display a preview version of the decoded image at a lower resolution. Progressively decoding the additional layers, the viewer can increase the spatial resolution of the image, up to the full resolution of the original image.

* Temporal scalability is defined as the representation of the same video in varying temporal resolutions or frame rates. The only diffenence between the spatial scalability and temporal scalability is that the spatially scalable codec uses spatial down-up sampling, where as the temporally scalable codec uses temporal down-up sampling.

* Frequency scalability represents a video frame in multiple layers by including different frequency components in each layer, with the base layer containing low-frequency components, and other layers containing increasingly higher-frequency components.

* The wavelet representation provides a multiresolution/multifrequency expression of a signal with localization in both time and frequency.

* Frequency scalability in MPEG2
MPEG2 lays the mode information, motion information, the first few DCT coefficients of each macroblock in the base layer, and including the remaining DCT coefficients in the enhancement layer. this is known as data partitioning.

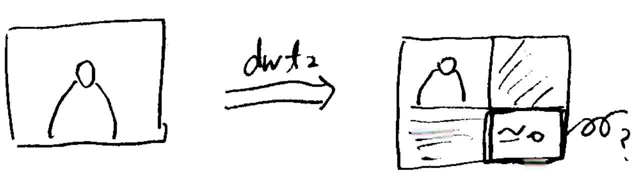
* MPEG 4
MPEG4 enhances the frame rate of a selected object such that it has a smoother motion than the remaining area
And MPEG4 has selected wavelet-based schemes as the basis for coding still texture and images

* JPEG 2000

Wavelet coding of still images in JPEG 2000 is based on the zero-tree-based coding concept known as embedded zero-tree wavelet. A DWT decomposes the input image into a set of subbands of varying resolutions. As scaning wavelet coefficients the coder encodes the wavelet coefficient. EZW further encodes coefficient values using a successive approximation quantization scheme.

* Further problem.

Currently we have various kinds of subband decomposition filters. In image compression, 'Daubechies' decomposition filter shows the result that most of energies of decomposed signals are on the horizontal and vertical band. It means that we can reduce the quantization bits on the diagonal band as following.



If we adaptively find the decomposition filter which depends on the source signal so that the energy in diagonal band is close to zero, we have huge advantage in Coding the signal. I think the possible candidates are Kahunen-Loeve Expansion or Singular Value Decomposition.