ECE 4270
Fundamentals of Digital Signal Processing

Lecture 10: Review of Sampling, Resolution & Observation Time, & Digital Processing of C-T Signals

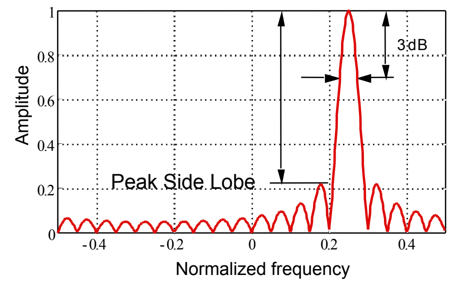
School of Electrical and Computer Engineering
Georgia Institute of Technology
Summer, 2004

A Fourier Transform Pair

$$x[n] = e^{j\omega_0 n}, \quad 0 \le n \le N-1$$

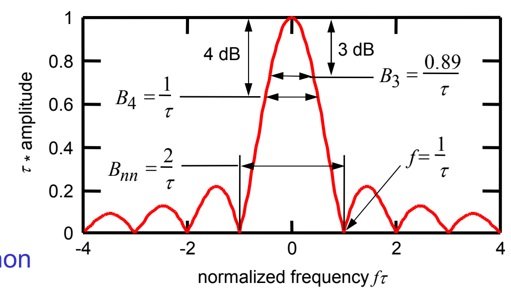
$$\Leftrightarrow \quad X(\omega) = \frac{\sin((\omega-\omega_0)N/2)}{\sin((\omega-\omega_0)/2)} e^{-j(\omega-\omega_0)(N-1)/2}$$

- Analog signal duration = NT sec
- Characteristics:
  - peak at ω_0
  - -13.2 dB sidelobes
  - mainlobe width in Hz
    - $0.89/NT$ @ -3 dB
    - $1/NT$ @ -4 dB
    - $2/NT$ null-to-null



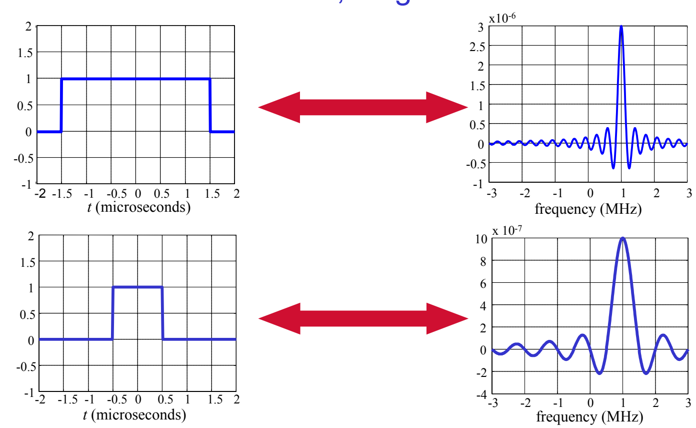
Reciprocal Spreading

- Product of signal observation time τ and "bandwidth" B is a constant k
  - $k = 0.89, 1,$ or 2 depending on definition of "bandwidth"
  - "fatter" in one domain implies "skinnier" in the other
  - Heisenberg uncertainty principle is another form of this same phenomenon



Reciprocal Spreading

- Note mainlobe width of sinc ~ $1/T$
  - "short in one domain, long in the other"



The Uncertainty Principle

- If we define the width of a signal in the signal or Fourier domain as its variance:

$$D_t^2 = \int_{-\infty}^{\infty} t^2 \left| f(t) \right|^2 dt, \quad D_\omega^2 = \int_{-\infty}^{\infty} \omega^2 \left| F(\omega) \right|^2 d\omega$$

- Then a direct consequence of the Fourier relation is the *Uncertainty Principle*:

$$D_t D_\omega \geq \sqrt{\frac{\pi}{2}}$$

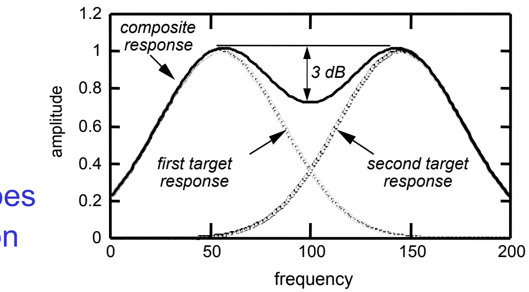
(*Gaussian waveform achieves equality*)

  – shorter in one domain implies longer in the other!

Heisenberg Uncertainty Principle is a consequence of Fourier relation between quantum mechanical descriptions of position and momentum

Resolution

- Resolution is the spacing (in range, frequency, angle, *etc*.) we must have in order to distinguish two signals
- Determined by response width in the relevant domain
- Two sinusoids are resolvable if the mainlobe of their Fourier transforms do not overlap too much
  – narrow mainlobes
  – long observation times

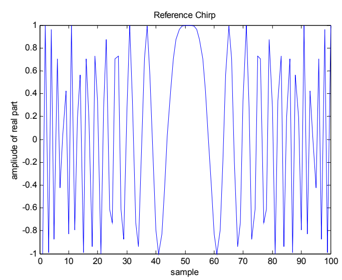


Correlation for Signal Detection

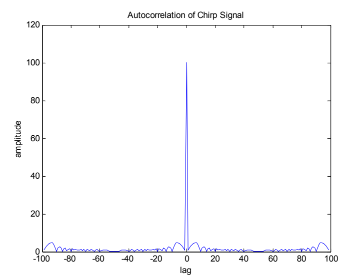
"Pulse Compression" Signals

- Signals can be designed to have a narrow, peaked autocorrelation
- Peak occurs when signal is exactly lined up with itself in the autocorrelation sum

$x[n]$

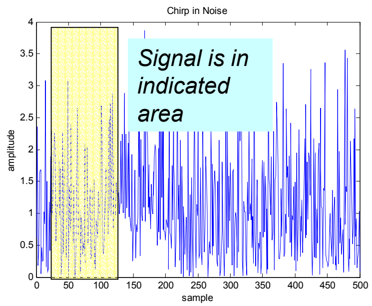


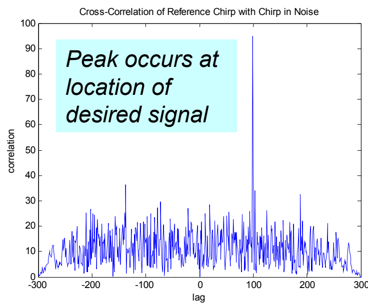
$c_{xx}[m]$



Signal Detection by Cross-Correlation

- Want to detect desired signal when it is buried in noise
- Cross-correlate noisy signal with clean copy of desired signal
  - Cross correlation is sum of autocorrelation of signal, and cross correlation of signal with noise
- Peak correlation occurs when the signal being tested is lined up exactly with the reference
- Correlating a noisy signal with a clean reference signal emphasizes the signal, not the noise:





Nyquist Theorem Again …

Nyquist Rate: The lowest sampling rate to permit an undistorted reconstruction of a **continuous time signal** from a sampled sequence; equal to twice the highest frequency in the continuous time signal

Nyquist Frequency: The highest frequency that is contained and represented in a **discrete time sequence**; equal to half of the sampling rate of the sequence

The Sampling Theorem

- A bandlimited signal with highest frequency less than the Nyquist frequency Ω_N can be reconstructed exactly from samples taken with sampling frequency:

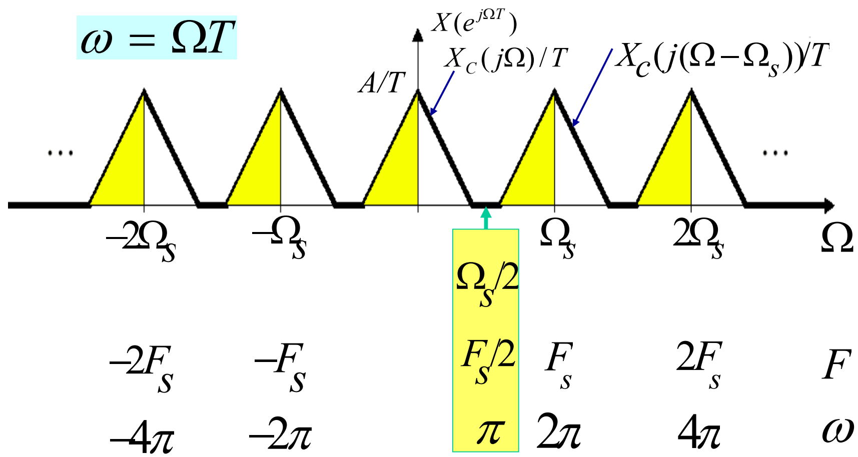
$$\Omega_S = \frac{2\pi}{T} = 2\Omega_N > 2\Omega_{\max} = \Omega_{NR}$$

- or in "cyclical frequency" units (Hz):

$$F_S = \frac{1}{T} = 2F_N > 2F_{\max} = F_{NR}$$

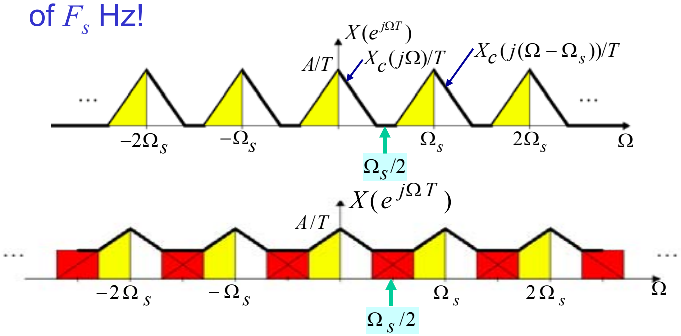
NR = Nyquist Rate

Mapping Between Analog and Digital Frequency



The Nyquist Theorem in the Other Direction

- If we sample a signal at a rate of $F_s = 1/T$ samples/second, we can “support” a signal having a highest frequency of $F_s/2$ Hz
  - Total two-sided spectral width of F_s Hz
  - The analog signal must “fit” into a spectrum width of F_s Hz!



Example: Digital Audio

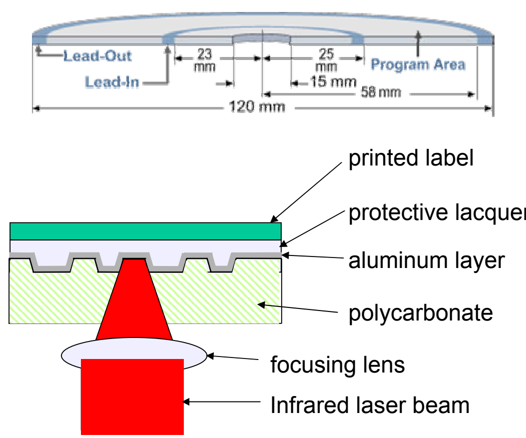
- The sampling rate for standard compact disc audio is 44.1 kHz @ 16 bits/sample
- Therefore this can support audio frequencies up to 44.1/2 kHz = 22.05 kHz
  - Consistent with the traditional high-fidelity audio frequency response requirement of 20 Hz to 20 kHz
- “Telephone quality” audio is sampled at 8 kHz or 16 kHz
  - Supports frequencies up to 4 kHz or 8 kHz
- Why 44.1 instead of 40 kHz (2x20 kHz) or 48 kHz (integer multiple of 8 and 16 kHz)
  - make it difficult to copy between formats?
  - Was convenient for early video-based recorder frame rates/sizes?
- DVD audio is up to 192 kHz @ 24 bits/sample

Compact Disc Format

Parameter	Value	Comments
Diameter:	12 (or 8) cm	Few 8 cm CDs are made
Thickness:	1.2 mm	Tolerance of +0.3, -0.1 mm
Width of pits:	0.5 microns	
Length of pits:	0.8 to 3	Depends on data stored
Depth of pits:	0.15 microns	
Scan velocity:	1.3 m/s	Tolerance +/- 0.1 m/s
Track pitch:	1.6 microns	Tolerance +/- 0.1 microns
Laser	770 to 830 nm	Typically 780 nm
Playing time:	74 m 44 s	Playing times can be longer
Number of	99 max	Can use indexes to subdivide
Modulation	EFM	8 to 14 bits plus 3 merging bits
Channel bit rate:	4.3218 Mb/s	Actual raw data rate

The raw data is read at 4.32 Mb/s, but after demodulation (17 bits become 8) and error correction the data rate is 1.41 Mb/s, which @16 b/sample means a sampling rate of 44.1 kHz. The total length of the helical track is about 5,800 meters.

Construction of Compact Disc



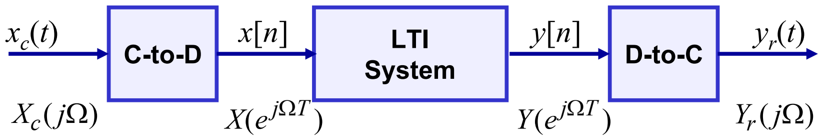
All audio CDs are played at a constant linear velocity (CLV) of 1.3 m/s. The angular velocity (rpm) will reduce from the lead-in to the lead-out by a factor of 58/23 = 2.52. This means that pits retain the same geometry wherever they are on the disc.

Digital Processing of Continuous-Time Signals

Idealized System (DSP Theory)

- A-to-D conversion --> C-to-D conversion
- Finite precision arithmetic --> real numbers
- D-to-A conversion --> D-to-C conversion

DT Filtering of CT Signals



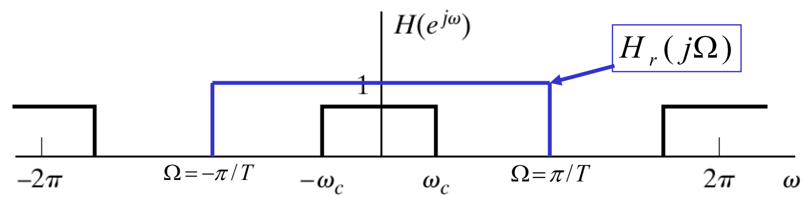
$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega Tn} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$Y(e^{j\Omega T}) = H(e^{j\Omega T})X(e^{j\Omega T})$$

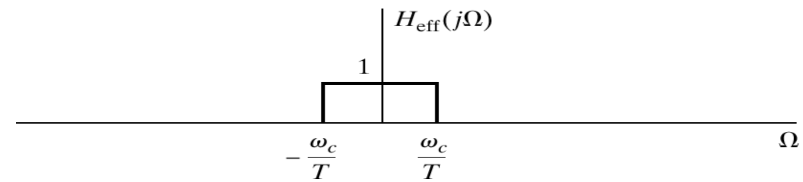
$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})$$

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

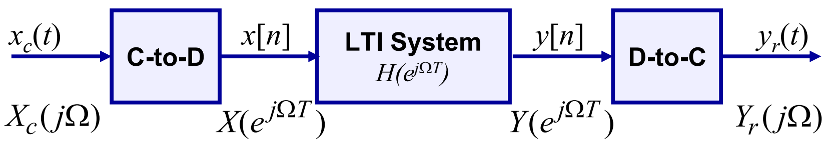
Effective Frequency Response



$$H_{\text{eff}}(j\Omega) = H(e^{j\Omega T}) \quad |\Omega| < \frac{\pi}{T}$$



DT Filtering of CT Signals



$$Y_r(j\Omega) = H(e^{j\Omega T})\left(\cancel{H_r(j\Omega)}\,1\,\frac{1}{T}\right)\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

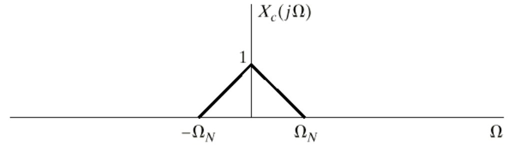
- If the input is bandlimited such that

$$X_c(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

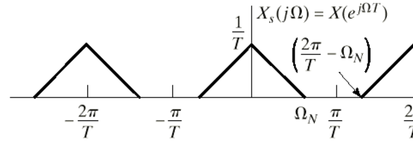
and $2\pi / T \geq 2\Omega_N$, then the overall input and output are related by

$$Y_r(j\Omega) = H(e^{j\Omega T})X_c(j\Omega)$$

D-T Linear Filtering of C-T Signals - 1



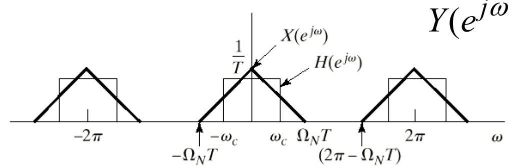
(a)



$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega + \frac{2\pi k}{T}\right)\right)$$

(b)

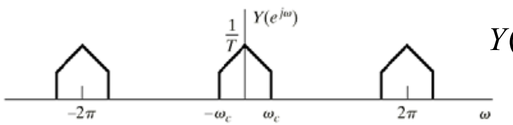
$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$



$$\omega = \Omega T$$

(c)

D-T Linear Filtering of C-T Signals - 2

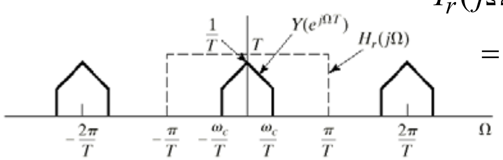


$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

(d)

$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})$$

$$= H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T})$$

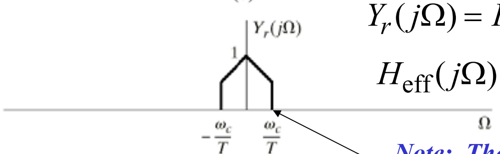


$$\Omega = \frac{\omega}{T}$$

(e)

$$Y_r(j\Omega) = H(e^{j\Omega T})H_r(j\Omega)\frac{1}{T}X_c(j\Omega)$$

$$H_{\text{eff}}(j\Omega) = H(e^{j\Omega T}) \quad |\Omega| < \pi / T$$



(f)

Note: The effective cutoff frequency depends on the sampling rate!

Another Example

- Difference equation:

$$y[n] = ay[n-1] + bx[n]$$

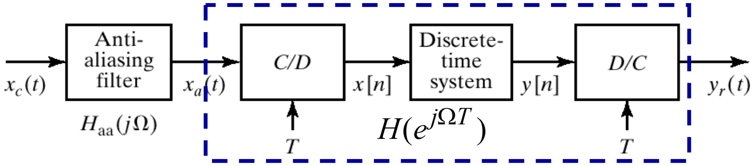
- Frequency response:

$$H(e^{j\omega}) = \frac{b}{1 - ae^{-j\omega}}$$

- Overall frequency response

$$H(j\Omega) = H(e^{j\Omega T}) = \frac{b}{1 - ae^{-j\Omega T}} \quad |\Omega| < \frac{\pi}{T}$$

Anti-Alias Pre-filtering



$$Y_r(j\Omega) = H(e^{j\Omega T})X_a(j\Omega) \text{ if } X_a(j\Omega)=0 \quad \text{for } |\Omega| \geq \Omega_N$$

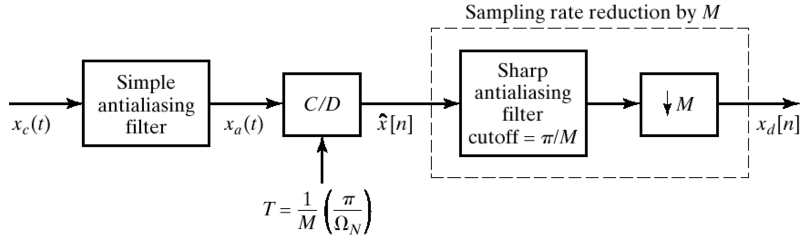
- What is the overall effective frequency response?

$$X_a(j\Omega) = H_{aa}(j\Omega)X_c(j\Omega)$$

$$\Rightarrow \quad X_a(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

$$Y_r(j\Omega) = \underbrace{H(e^{j\Omega T})H_{aa}(j\Omega)}_{H_{\text{eff}}(j\Omega)}X_c(j\Omega)$$

Oversampling Eases Filtering - I



$$X_a(j\Omega) = H_{aa}(j\Omega)X_c(j\Omega)$$

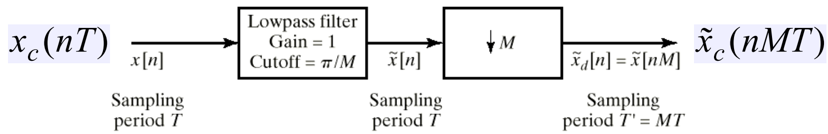
$$\text{Choose } \; H_{aa}(j\Omega) = 0 \quad \text{for } |\Omega| \geq M\Omega_N$$

$$\Rightarrow X_a(j\Omega) = 0 \quad \text{for } |\Omega| \geq M\Omega_N$$

Decimation: What's Going On

- "Decimation" refers to lowering the sampling rate
  - Fewer samples/second
- Therefore we will reduce the bandwidth that can be supported without aliasing
- Therefore we have to ensure that the analog signal bandwidth will "fit" into the new, *reduced* DTFT spectrum width before we decimate it
- So we lowpass filter it before decimating to make sure its bandwidth is not too large

Decimation - I



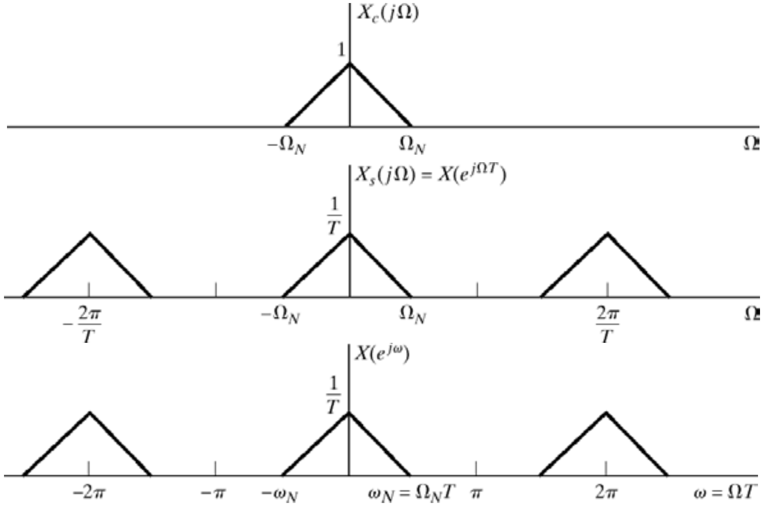
$$x[n] \Leftrightarrow X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$\tilde{x}[n] \quad \Leftrightarrow \quad \tilde{X}(e^{j\Omega T}) = H(e^{j\Omega T})X(e^{j\Omega T})$$

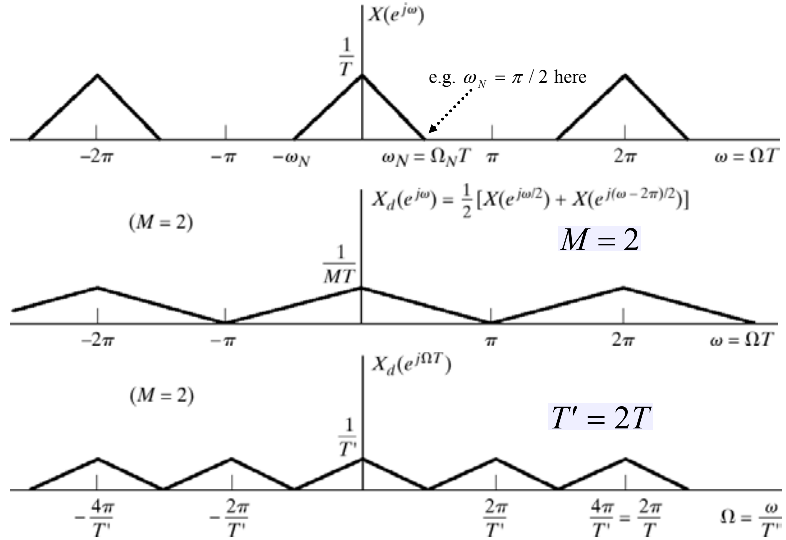
$$\tilde{x}_d[n] \Leftrightarrow \tilde{X}_d(e^{j\Omega MT}) = \frac{1}{MT}\sum_{k=-\infty}^{\infty} \tilde{X}_c\left(j(\Omega - \frac{2\pi k}{MT})\right)$$

$$= \frac{1}{M}\sum_{r=0}^{M-1}\tilde{X}(e^{j(\Omega T - 2\pi r)/M}) = \frac{1}{M}\sum_{r=0}^{M-1}\tilde{X}(e^{j(\omega - 2\pi r)/M})$$

Decimation - II

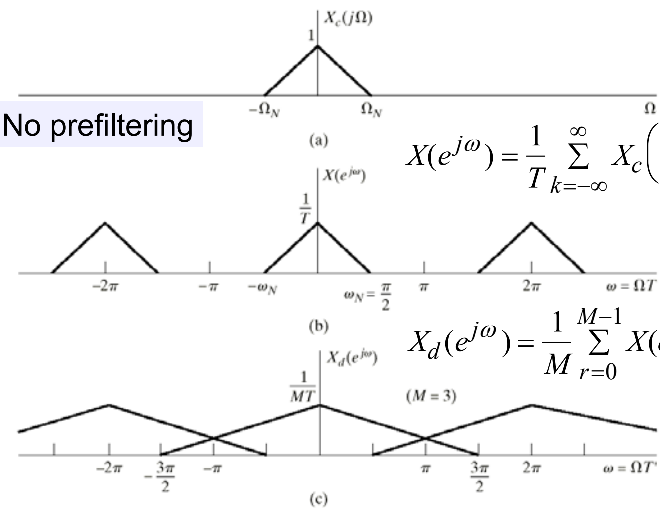


Decimation - III



Decimation - III

No prefiltering



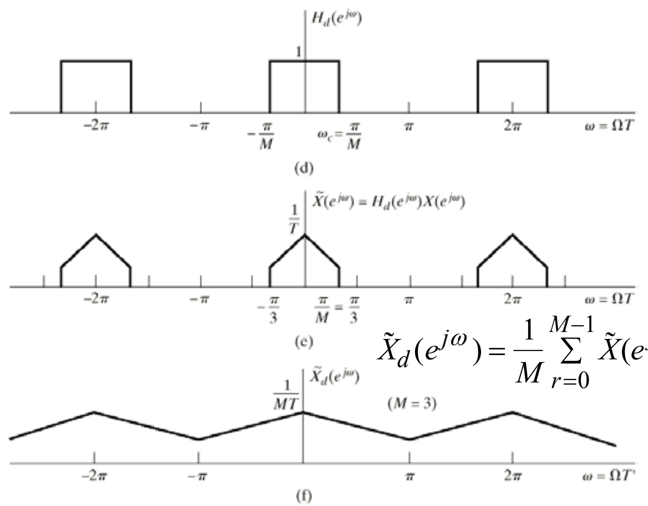
$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

$M = 3$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{r=0}^{M-1} X(e^{j(\omega-2\pi r)/M})$$

$T' = 3T$

Decimation - IV



$M = 3$

$$\tilde{X}_d(e^{j\omega}) = \frac{1}{M}\sum_{r=0}^{M-1} \tilde{X}(e^{j(\omega-2\pi r)/M})$$

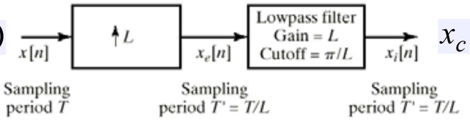
$T' = 3T$

Interpolation: What's Going On

- We want to increase the sampling rate
  - More samples/second
- This will increase the analog bandwidth that will "fit" into the spectrum, so we don't have the problem of too much analog bandwidth as we did in decimation
- To increase the number of samples by M, we insert M-1 zeroes between each existing sample
  - Causes a spectrum replication effect
  - We need to replace the zero values with interpolated values
- Apply a digital filter to get rid of the spectrum copies
  - Effectively interpolates between the original samples
  - Similar to how a reconstruction filter interpolates discrete samples back to a continuous-time signal

Interpolation - I

$x_c(nT)$



$x_c(nT/L)$

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL] = \begin{cases} x[n/L], & n = 0, \pm L, \dots \\ 0, & \text{otherwise} \end{cases}$$

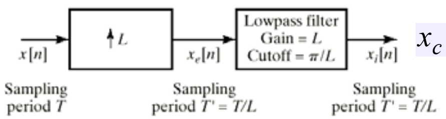
$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin[\pi(n-kL)/L]}{[\pi(n-kL)/L]} \quad \text{or since}$$

$$x_c(t) = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin[\pi(t-kT)/T]}{[\pi(t-kT)/T]}$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin[\pi(nT/L-kT)/T]}{[\pi(nT/L-kT)/T]} = x_c(nT/L)$$

Interpolation - II

$x_c(nT)$



$x_c(nT/L)$

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]e^{-j\omega n}$$

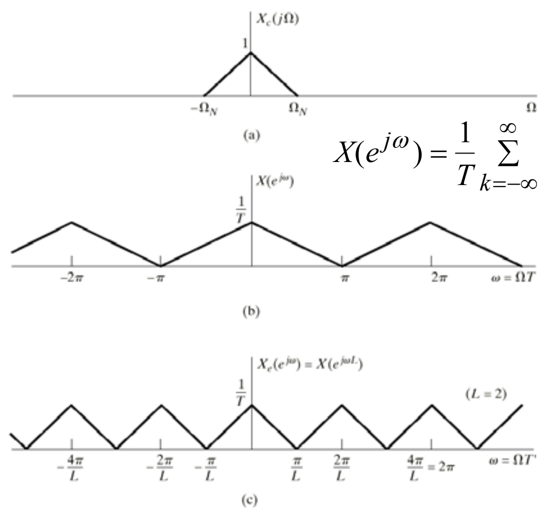
$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega Lk} = X(e^{j\omega L})$$

$$X_e(e^{j\Omega T/L}) = X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$X_i(e^{j\Omega T/L}) = H_i(e^{j\Omega T/L})X_e(e^{j\Omega T/L})$$

$$= \frac{1}{(T/L)}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T/L}\right)\right)$$

Interpolation - III

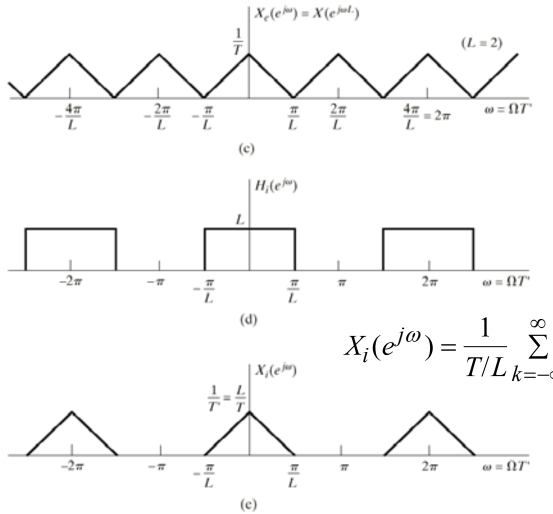


$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

$L = 2$

$T' = T/2$

Interpolation - IV



$$X_i(e^{j\omega}) = \frac{1}{T/L} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T/L} - \frac{2\pi k}{T/L}\right)\right)$$

Other Processing Schemes

- Temporal reversal

$$y[n] = x[-n]$$

- Amplitude Modulation or translation of spectrum

$$y[n] = x[n] \bullet A\cos(n\omega_0 + \phi)$$

- Spectrum flipping

$$y[n] = x[n] \bullet (-1)^n$$

- Echo & Reverberation

$$y[n] = x[n] + x[n - n_0]$$

$$y[n] = x[n] + \left(\sum_{m=-\infty}^{\infty} x[n - n_0 - m]h[m]\right)$$