ECE 4270

Fundamentals of Digital Signal Processing

Lecture 14:

Frequency Response of Rational LTI Systems, Digital Filter Structures

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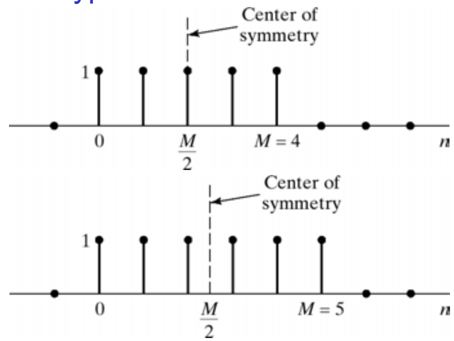
Generalized Linear Phase FIR Systems

$$h[M-n]=h[n] \quad 0\le n\le M$$

$$H(e^{j\omega})=A_e(e^{j\omega})e^{-j\omega M/2}, \quad A_e(e^{j\omega})=A_e(e^{-j\omega})$$

- Types I & II: $H(z^{-1})=z^{M}H(z)$



Type I:
M even
integer delay

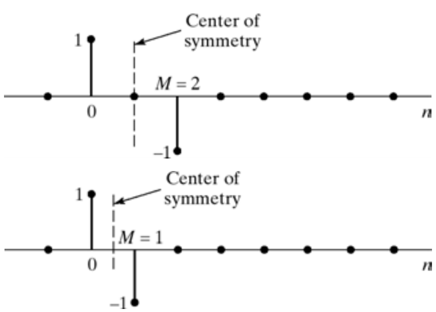
Type II:
M odd
half sample delay

Generalized Linear Phase FIR Systems

$$h[M-n]=-h[n] \quad 0\le n\le M$$

$$H(e^{j\omega})=jA_o(e^{j\omega})e^{-j\omega M/2}, \quad A_o(e^{j\omega})=-A_o(e^{-j\omega})$$

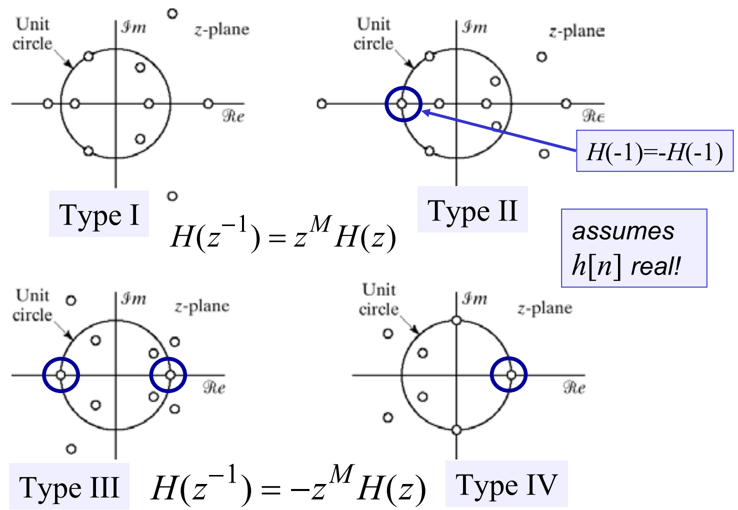
- Types III & IV: $H(z^{-1})=-z^{M}H(z)$



Type III:
M even
integer delay

Type IV:
M odd
half sample delay

Zero Locations for FIR Linear Phase



assumes $h[n]$ real!

Minimum Phase, Non-Minimum Phase, And Maximum Phase Sequences

Minimum Phase

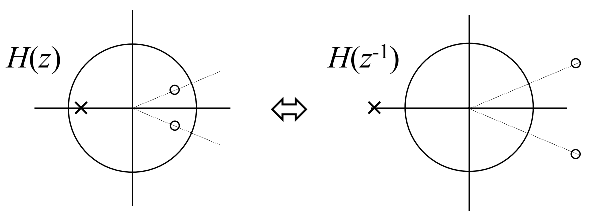
- A Linear Time-Invariant (LTI) filter $H(z) = B(z)/A(z)$ is said to be *minimum phase* if all its poles and zeros are inside the unit circle $|z| = 1$.
- A minimum phase filter has to be *causal*; noncausal terms in the impulse response correspond to poles at infinity (e.g., $H(z)=z$ has zero at 0 and pole at ∞).
- A filter is minimum phase if both the numerator and denominator of its transfer function are *minimum phase polynomials* in z^{-1}.
- A polynomial of the form

$$B(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}$$
$$= b_0 (1 - c_1 z^{-1})(1 - c_2 z^{-1}) \cdots (1 - c_M z^{-1})$$

  is said to be *minimum phase* if all of its roots c_i are inside the unit circle, i.e., $|c_i|<1$.
- A signal is *minimum phase* if its z-transform is minimum phase. Otherwise, it is non-minimum phase.

Maximum Phase & Mixed Phase

- A polynomial is said to be *maximum phase* if all roots of the polynomial are outside the unit circle.
- A polynomial is said to be *mixed phase* if some of its roots are inside the unit circle and some are outside; when its roots are ON the unit circle, it's called *marginally minimum/maximum phase* correspondingly.
- If $H(z)$ is minimum phase, then $H(z^{-1})$ is maximum phase and vice versa, because substituting z with z^{-1} would "flip" the poles and zeros in and out of the unit circle.



Minimum Phase & Maximum Decaying Sequence

- A sequence $1, c, 0, 0, \ldots$ has a z-transform $H(z) = 1 + cz^{-1}$
- It is minimum phase if $|c| < 1$, meaning, it's a decaying sequence.
- Many signals have identical power spectral density

$$H(z)H^*(z)\big|_{z=e^{j\omega}} = \prod_{i=1}^{M} (1 - c_i e^{-j\omega})(1 - c_i^* e^{j\omega})$$

  (e.g., any H that has the collection of either (c_i, c_i^*) or $(c_i^{-1}, (c_i^*)^{-1})$ as its root pairs)
- The minimum-phase signal $h_{\text{m-p}}(n)$ has the *fastest decay* in the sense that

$$\sum_{n=0}^{K} |h_{\text{m-p}}(n)|^2 \geq \sum_{n=0}^{K} |h_i(n)|^2 \quad K = 0, 1, 2, \cdots$$

  Thus, minimum-phase signals are *maximally concentrated toward time* 0 among all causal signals $h_i(n)$ with a given magnitude spectrum.

All-Pass Systems

First order all-pass filter $$H_{ap}(z)=\frac{z^{-1}-a^*}{1-az^{-1}}$$

$$H_{ap}(e^{j\omega})=\frac{e^{-j\omega}-a^*}{1-ae^{-j\omega}}=e^{-j\omega}\frac{1-a^*e^{j\omega}}{1-ae^{-j\omega}} \qquad |H_{ap}(e^{j\omega})|=1$$

$$20\log_{10}|H_{ap}(e^{j\omega})|=0 \qquad a=re^{j\theta}$$

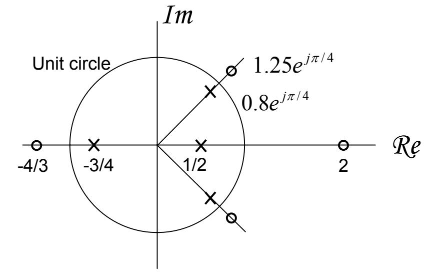
$$\angle H(e^{j\omega})=\angle\frac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}=-\omega-2\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$$

$$\mathrm{grd}\left[\frac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right]=\frac{1-r^2}{1+r^2-2r\cos(\omega-\theta)}=\frac{1-r^2}{|1-re^{j\theta}e^{-j\omega}|^2}$$

Higher Order All-Pass Systems

Typical pole-zero pattern of an all-pass system

- If a is a pole, then $(a^*)^{-1}$ must be a zero
- If a is a zero, then $(a^*)^{-1}$ must be a pole



All-pass filters are used to alter the phase property of a signal or a system, for example:

- compensate for phase or group delay distortion;
- change the peak to rms ratio of a signal without changing its log magnitude spectrum
- "construct" minimum-phase systems

Stable All-Pass Systems has Positive GRD

First order all-pass filter $$H_{ap}(z)=\frac{z^{-1}-a^*}{1-az^{-1}}$$

$$a=re^{j\theta} \qquad |H_{ap}(e^{j\omega})|=1$$

For a stable and causal all-pass system,

$r<1$ (i.e., poles are within the unit circle)

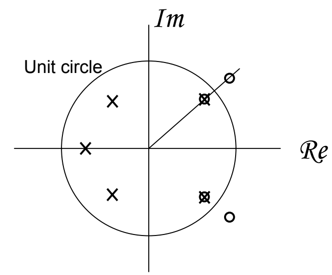
$$\mathrm{grd}\left[\frac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right]=\frac{1-r^2}{1+r^2-2r\cos(\omega-\theta)}=\frac{1-r^2}{|1-re^{j\theta}e^{-j\omega}|^2}>0$$

Minimum-Phase & All-Pass Decomposition

Any rational system can be expressed as

$$H(z)=H_{m-p}(z)H_{a-p}(z)$$

Assume $H(z)$ has one zero outside the unit circle at $z=1/c^*$, $|c|<1$



$$H(z)=H_1(z)(z^{-1}-c^*)=H_1(z)(1-cz^{-1})\frac{(z^{-1}-c^*)}{(1-cz^{-1})}$$

$H_1(z)(1-cz^{-1})$ is minimum phase because $|c|<1$

Therefore, $H(z)=H_{m-p}(z)H_{a-p}(z)$

Minimum Phase & Minimum GRD

$$H(z) = H_{m-p}(z)H_{a-p}(z)$$

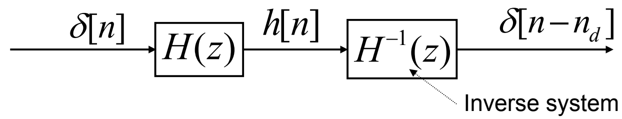
$$\angle H(z) = \angle H_{m-p}(z) + \angle H_{a-p}(z)$$

$$\text{grd}[H(z)] = \text{grd}[H_{m-p}(z)] + \text{grd}[H_{a-p}(z)]$$

$$\text{grd}[H_{a-p}(z)] \geq 0$$

Therefore, of all the $H(z)$ that have the same magnitude spectrum, the corresponding minimum phase system has the minimum group delay.

Inverse Systems



- An inverse system produces an impulse as output when driven by the impulse response of the original system.
- In practice, the output impulse may have a relative delay with respect to the original impulse.

Example:

$$H(z) = \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}} \qquad H_{inv}(z) = H^{-1}(z) = 1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}$$

$$h[n] = \frac{r^n \sin[\theta(n+1)]}{\sin\theta} u[n] \qquad h_{inv}[n] = \delta[n] - 2r\cos\theta\, \delta[n-1] + r^2\delta[n-2]$$

Digital Filters

- General N^{th}-order difference equation:

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

- System function:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = A\frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- There is a direct correspondence between the difference equation and the system function when the numerator and denominator are written as polynomials in z^{-1}.

Direct Form I Implementation

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \underbrace{\left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}}\right)}_{\text{poles}} \underbrace{\sum_{k=0}^{M} b_k z^{-k}}_{\text{zeros}}$$

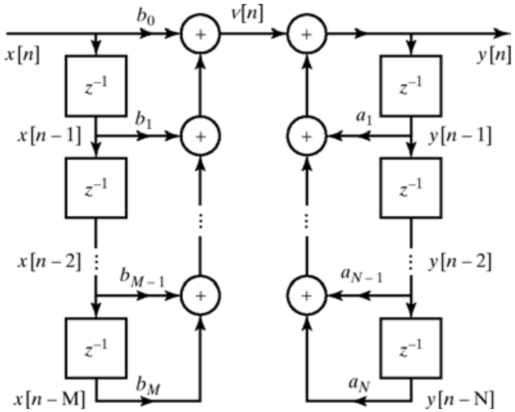
$$Y(z) = H(z)X(z)$$

$$v[n] = \sum_{k=0}^{M} b_k x[n-k] \quad \text{(zeros)}$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + v[n] \quad \text{(poles)}$$

Block Diagram of IIR Direct Form I

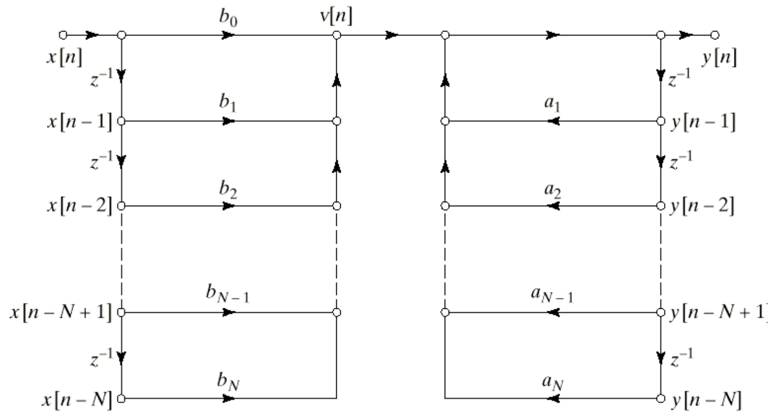
$$v[n] = \sum_{k=1}^{M} b_k x[n-k] \qquad y[n] = \sum_{k=1}^{N} a_k y[n-k] + v[n]$$



(zeros) (poles)

Signal Flow Graph IIR Direct Form I

$$v[n] = \sum_{k=1}^{M} b_k x[n-k] \qquad y[n] = \sum_{k=1}^{N} a_k y[n-k] + v[n]$$



(zeros) (poles)

Direct Form II Implementation

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \underbrace{\sum_{k=0}^{M} b_k z^{-k}}_{\text{zeros}} \underbrace{\left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right)}_{\text{poles}}$$

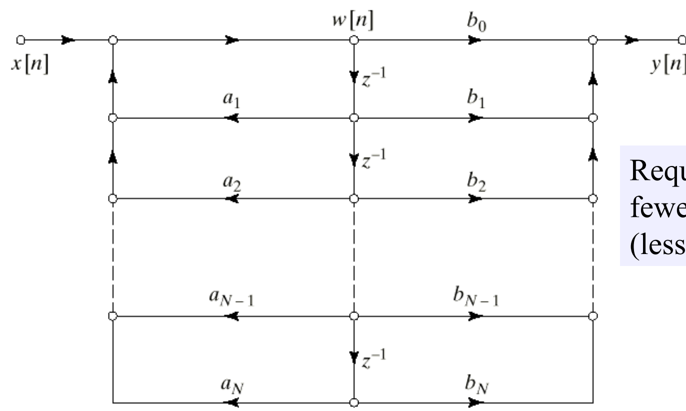
$$Y(z) = H(z)X(z)$$

$$w[n] = \sum_{k=1}^{N} a_k w[n-k] + x[n] \quad \text{(poles)}$$

$$y[n] = \sum_{k=0}^{M} b_k w[n-k] \quad \text{(zeros)}$$

IIR Direct Form II

$$w[n] = \sum_{k=1}^{N} a_k w[n-k] + x[n] \qquad y[n] = \sum_{k=1}^{M} b_k w[n-k]$$



(poles) (zeros)

Requires fewer delays (less memory)