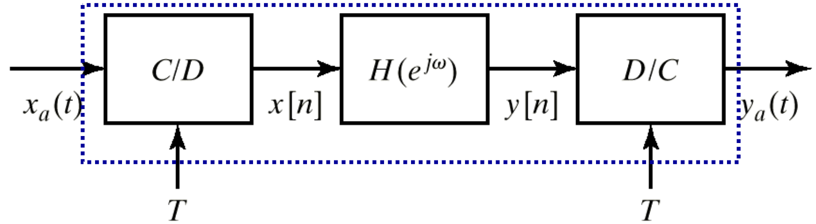
ECE 4270

Fundamentals of Digital Signal Processing

Lecture 16: IIR Filter Design by Transformations, Window Design of FIR Filters

School of Electrical and Computer Engineering
Georgia Institute of Technology
Summer, 2004

Digital Filtering

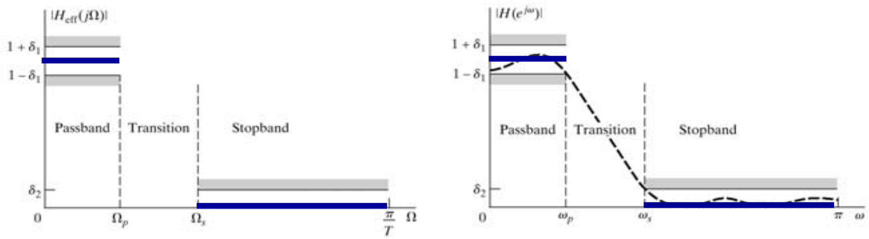


$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi / T \\ 0, & |\Omega| > \pi / T \end{cases}$$

$\omega = \Omega T$

$$H(e^{j\omega}) = H_{\text{eff}}\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

Setting the Specifications



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$\omega = \Omega T$

$$H(e^{j\omega}) = H_{\text{eff}}\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

A Design Example

- The C-T specifications are ($1/T$=2000 Hz):

$$0.99 \le |H_{\text{eff}}(j\Omega)| \le 1.01, \qquad |\Omega| \le 2\pi(400)$$
$$|H_{\text{eff}}(j\Omega)| \le 0.001, \quad 2\pi(600) \le |\Omega| \le 2\pi(1000)$$

i.e., $\Omega_p = 2\pi(400)$ and $\Omega_s = 2\pi(600)$.

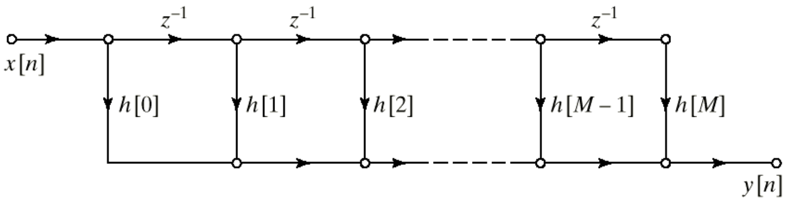
- The corresponding D-T specifications are:

$$0.99 \le |H(e^{j\omega})| \le 1.01, \qquad |\omega| \le 0.4\pi$$
$$|H(e^{j\omega})| \le 0.001, \quad 0.6\pi \le |\omega| \le \pi$$

i.e., $\omega_p = \Omega_p T = 0.4\pi$ and $\omega_s = \Omega_s T = 0.6\pi$.

FIR Filter Design

- Find the impulse response so that the filter meets a set of specifications on the frequency response.

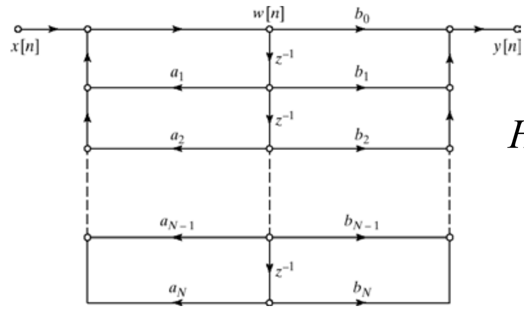


$$H(z)=\sum_{n=0}^{M} h[n]z^{-n} \qquad H(e^{j\omega})=H(z)\big|_{z=e^{j\omega}}$$

Polynomial approximation

IIR Filter Design

- Find $B(z)$ and $A(z)$ so that the filter meets the specifications on the frequency response.



$$H(e^{j\omega})=H(z)\big|_{z=e^{j\omega}}$$

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}=\frac{B(z)}{A(z)}$$

Rational function approximation

Aside: Reminder of Some Formulas for Analog Fourier Transforms and Frequency Response

Analogies

Continuous	Discrete
$x_c(t)$	$x[n]$
$X_c(s)=\int_{-\infty}^{\infty} x_c(t)e^{-st}dt$	$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$
$s=\sigma+j\Omega$	$z=re^{j\omega}$
$X_c(j\Omega)=\int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}dt$	$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
$=X_c(s)\big\|_{s=j\Omega}$	$=X(z)\big\|_{z=e^{j\omega}}$

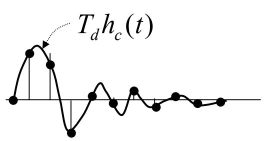
Return to IIR Filter Design …

Filter Design by Impulse Invariance

- Design D-T IIR filters from C-T filters
- Obtain D-T system impulse response by sampling the impulse response of a C-T system; D-T system frequency response determined by the C-T system frequency response
- Only works with band-limited systems; does not apply to high pass C-T systems because they are not band-limited – sampling theorem does not apply
- Due to possible aliasing, stop band usually needs to be over designed.

Impulse Invariance Design Technique

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right)$$



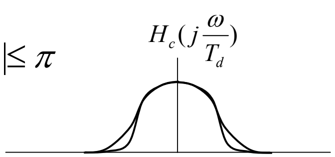
$$h[n] = T_d h_c(nT_d)$$

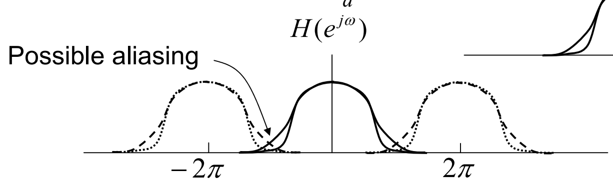
If the C-T filter is band-limited,

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T_d,$$

then

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right), \quad |\omega| \leq \pi$$





Bilinear Transformation

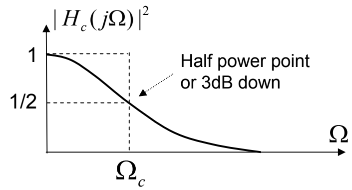
- C-T system design techniques are very advanced; system response defined in the s-plane, i.e. $H_c(s)$
- Use bilinear transformation between s and z, to map the $j\Omega$-axis in the s-plane to one full revolution of the unit circle in the z-plane.
- Incurs frequency warping and only useful in systems with piecewise-constant magnitude response, if conformance to the response characteristics is required in the design.
- Cannot use bilinear transformation to design D-T differentiators due to frequency warping

Common Continuous Time IIR Filters

1. Butterworth Lowpass Filters:

– magnitude response is maximally flat in the passband

$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}$$



Other Properties:

– First (2N-1) derivatives at DC are zero

– Magnitude response is monotonic in both pass band and stop band

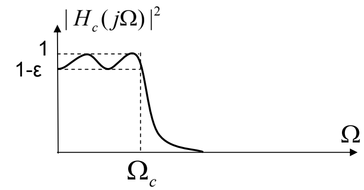
– The higher the order, the sharper the slope

Common Continuous Time IIR Filters

2. Chebyshev Filters:

– magnitude response is equi-ripple in pass band and monotone in stop band or vice versa

– Approximation error distributed uniformly across the equi-ripple band



$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^2(\Omega/\Omega_c)}$$

where $V_N(x)$ is the Nth - order Chebyshev polynomial,

$$V_N(x) = \cos(N\cos^{-1}x) \quad \text{and} \quad V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x)$$

Common Continuous Time IIR Filters

3. Elliptic Filters:

– Equi-ripple in both pass band and stop band;

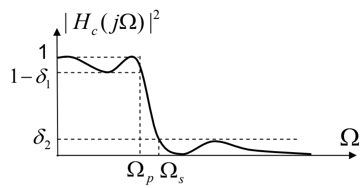
– approximation error distributed uniformly across both bands – best and achieved with lowest order

– For a given N, pass band frequency and error bounds, it has the smallest transition band ($\Omega_s - \Omega_p$)

– Has both poles and zeros (zeros on the Im axis of the s-plane)

$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N^2(\Omega)}$$

where $U_N(\Omega)$ is a Jacobian elliptic function



Bilinear Transformation

- We simply transform an analog filter $H_c(s)$ into a digital filter $H(z)$ with the complex mapping

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right); \text{ i.e., } H(z) = H_c(s)\Big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}.$$

$$z = \left(\frac{1+sT_d/2}{1-sT_d/2}\right) \Rightarrow re^{j\omega} = \left(\frac{1+\sigma T_d/2 + j\Omega T_d/2}{1-\sigma T_d/2 - j\Omega T_d/2}\right)$$

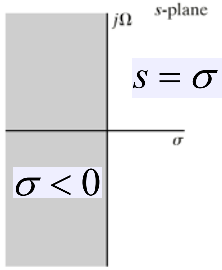
$$\sigma = 0 \Rightarrow r = 1 \text{ and } \omega = 2\arctan(\Omega T_d/2)$$

$$\sigma < 0 \Rightarrow r < 1 \text{ and } \sigma > 0 \Rightarrow r > 1$$

Bilinear Transformation

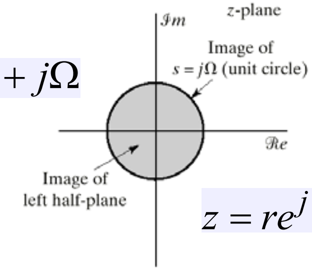
$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$z = \left(\frac{1 + sT_d/2}{1 - sT_d/2}\right)$$



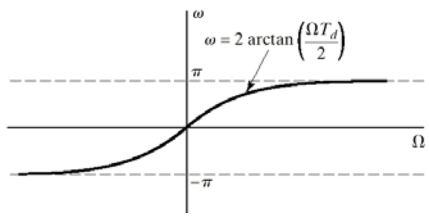
$s = \sigma + j\Omega$

$\sigma < 0$



$z = re^{j\omega}$

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right]$$



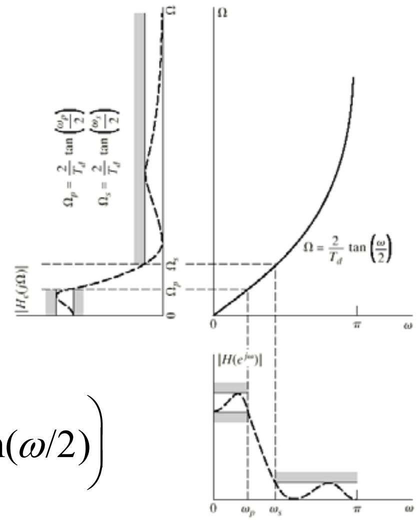
Bilinear Transformation of $H_c(j\Omega)$

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right]$$

$$\omega = 2\arctan(\Omega T_d/2)$$

$$\Omega = \frac{2}{T_d}\tan(\omega/2)$$

$$H(e^{j\omega}) = H_c\left(j\frac{2}{T_d}\tan(\omega/2)\right)$$



Simple (Butterworth) Example

$$H_c(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \Rightarrow |H_c(j\Omega)|^2 = \frac{1}{1 + \Omega^4}$$

- The digital filter has system function

$$H(z) = \frac{1}{\left(\frac{2}{T_d}\frac{1 - z^{-1}}{1 + z^{-1}}\right)^2 + \sqrt{2}\left(\frac{2}{T_d}\frac{1 - z^{-1}}{1 + z^{-1}}\right) + 1}$$

$$|H(e^{j\omega})|^2 = \left|H_c\left(j\frac{2}{T_d}\tan(\omega/2)\right)\right|^2 = \frac{1}{1 + \left(\frac{2}{T_d}\tan(\omega/2)\right)^4}$$

A Design Example

- The D-T specifications are:

$0.99 \le |H(e^{j\omega})| \le 1.01, \quad |\omega| \le 0.4\pi$ Pass band

$|H(e^{j\omega})| \le 0.001, \quad 0.6\pi \le |\omega| \le \pi$ Stop band

i.e., $\omega_p = \Omega_p T = 0.4\pi$ and $\omega_s = \Omega_s T = 0.6\pi$.

- The continuous-time prototype filter $H_c(j\Omega)$ must satisfy:

$$0.99 \le |H_c(j\Omega)| \le 1.01, \quad |\Omega| \le \frac{2}{T_d}\tan(0.4\pi/2)$$

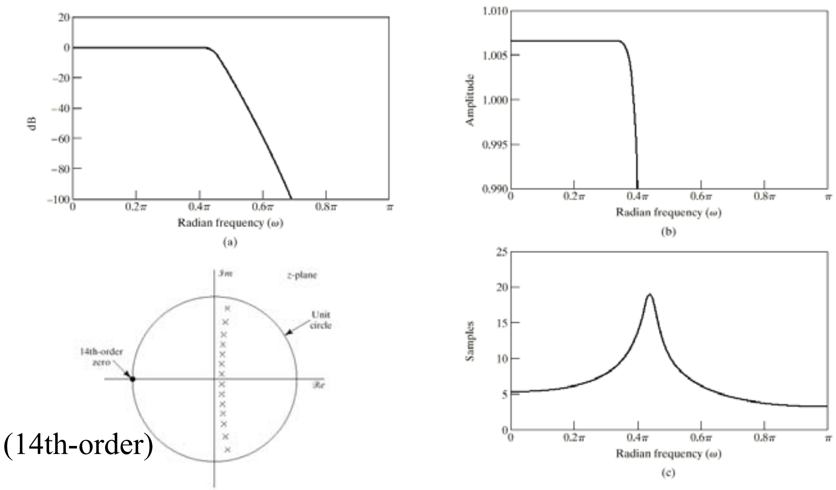
$$|H_c(j\Omega)| \le 0.001, \quad \frac{2}{T_d}\tan(0.6\pi/2) \le |\Omega| < \infty$$

Butterworth Approx. in MATLAB

```
»[N,wp]=buttord(.4,.6,-20*log10(.99),-20*log10(.001))
N = 14
wp = 0.44490626110897

» [b,a]=butter(N,wp)
b =
 Columns 1 through 7
  0.0001   0.0011   0.0071   0.0284   0.0782   0.1563   0.2345
 Columns 8 through 14
  0.2680   0.2345   0.1563   0.0782   0.0284   0.0071   0.0011
 Column 15
  0.0001
a =
 Columns 1 through 7
  1.0000  -1.5395   2.9473  -2.8363   2.7428  -1.7703   1.0616
 Columns 8 through 14
 -0.4647   0.1794  -0.0516   0.0123  -0.0021   0.0003  -0.0000
 Column 15
  0.0000
```

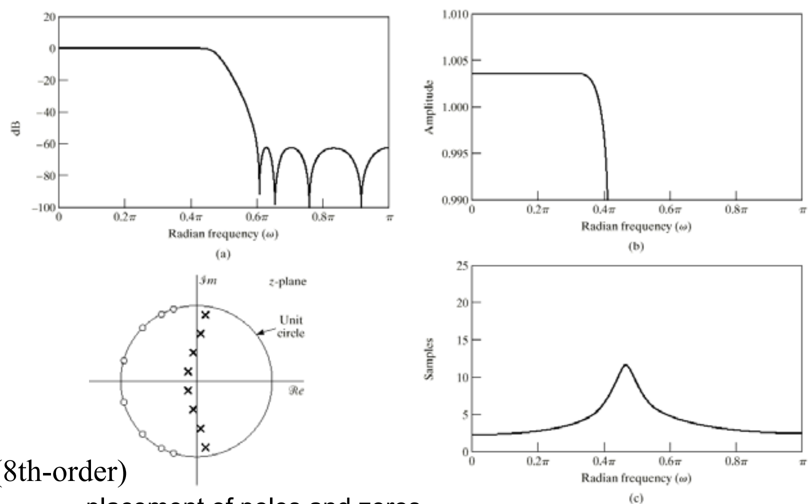

Butterworth Approximation



(14th-order)

Remark: find placement of 7 pole-pair to best match the spec.

Chebyshev Approximation



(8th-order)

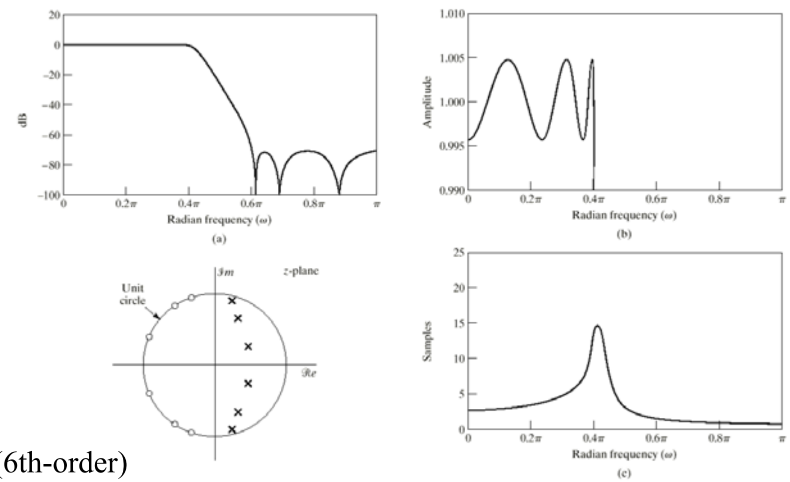
placement of poles and zeros

Elliptic Approximation in MATLAB

```
»[N,wp]=ellipord(.4,.6,-20*log10(.99),-20*log10(.001))
N =
    6
wp =
   0.4000

»[b,a]=ellip(N,-20*log10(.99),-20*log10(.001),wp)
b =
   0.0208   0.0590   0.1068   0.1265   0.1068   0.0590   0.0208
a =
   1.0000  -1.9585   2.8916  -2.5155   1.5627  -0.5906   0.1148
```

Elliptic Approximation

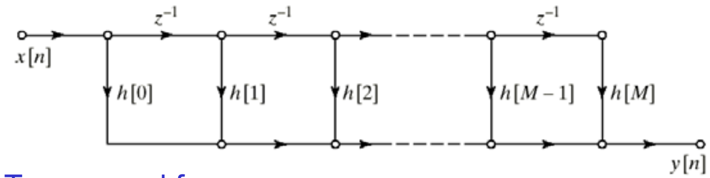


(6th-order)

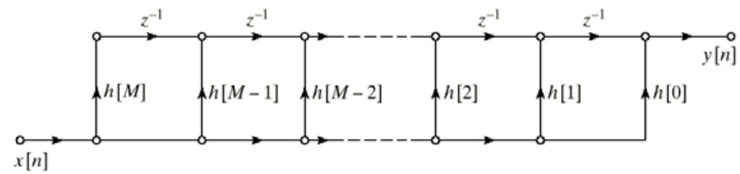
FIR Filter Structures

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] \qquad H(z) = \sum_{k=0}^{M} h[k]z^{-n}$$

- Direct form:



- Transposed form



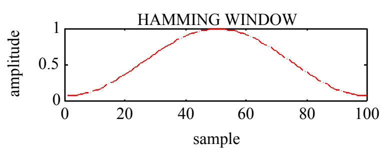
Window Method for Design of FIR Filters

- *Step 1*. Compute analytically the ideal impulse response, usually by inverse Fourier transformation of an ideal frequency response
  - example: ideal low pass filter, cutoff f_0, unity gain, has ideal impulse response of infinite duration: $h_{\text{ideal}}[n] = \dfrac{\sin(2\pi f_0 n)}{\pi n}$
- Step 2. Obtain a finite-duration approximation by windowing the ideal response, *i.e.* multiplying by a tapered, finite-length function

$$h[n] = h_{\text{ideal}}[n] \bullet w[n]$$

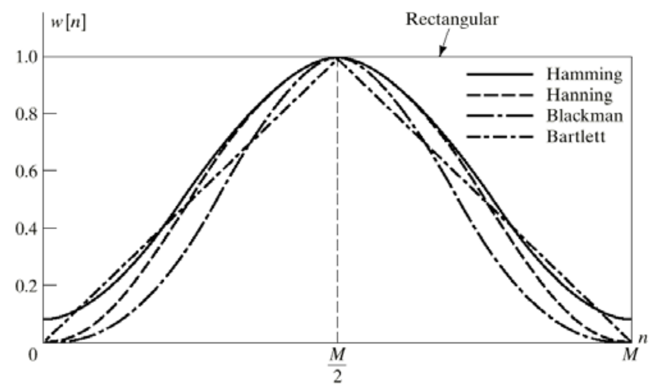
- example: Hamming window:

$$w[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \le n \le N-1$$



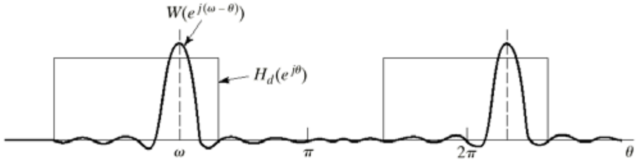
  - the window both limits the impulse response duration and controls the frequency response sidelobes
  - for a given sidelobe level, sharpness of cutoff increases with N

Some Common Windows

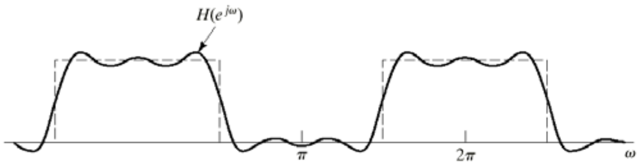


Effect of the Window Method on Frequency Response

$$h[n] = w[n]h_d[n] \quad \Leftrightarrow \quad H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$



(a)



(b)

Illustration of Window Design of Half-Band Lowpass Filter