ECE 4270

Fundamentals of Digital Signal Processing

Lecture 17:
Window Design of FIR Filters

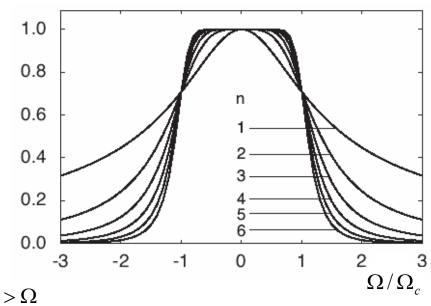
School of Electrical and Computer Engineering
Georgia Institute of Technology
Summer, 2004

Review of IIR Filter Design by Bilinear Transformation

More on Butterworth Filters

$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}$$

$|H_c(j\Omega)|^2$



$$|H_c(j\Omega)|^2 = \frac{\Omega_c^{2N}}{\Omega_c^{2N}+\Omega^{2N}}$$

$$\underset{N\to\infty}{\to} \frac{\Omega_c^{2N}}{(\max\{\Omega_c,\Omega\})^{2N}} \to \begin{cases} 1 & \text{if } \Omega_c > \Omega \\ 0 & \text{if } \Omega_c < \Omega \end{cases}$$

$\int_0^\infty |H_c(j\Omega)|^2 \, d\Omega =$ 1.571, 1.111, 1.047, 1.026, 1.017 and 1.012. for *N* = 1, 2, . . . , 6, respectively

More on Butterworth Filter

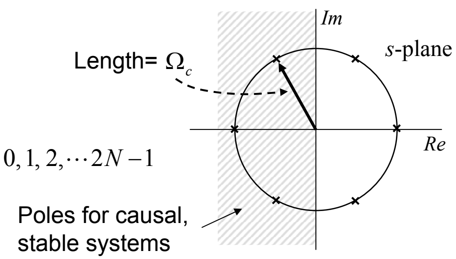
$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}} \Leftarrow H_c(s)H_c(-s) = \frac{1}{1+(s/j\Omega_c)^{2N}}$$

Poles are at the roots of $1+(s_k/j\Omega_c)^{2N} = 0$

$$s_k = (-1)^{1/2N}(j\Omega_c)$$

$$= \Omega_c \exp\left\{\frac{j\pi}{2N}(2k-1)\right\}\exp\left\{\frac{j\pi}{2}\right\}$$

$$= \Omega_c \exp\left\{\frac{j\pi}{2N}(2k+N-1)\right\} \quad k = 0, 1, 2, \cdots 2N-1$$



$$H_c(s) = \frac{1}{\prod (s-s_k)(s-s_k^*)} \quad \text{where } s_k^* \text{ and } s_k^* \text{ are on the left plane}$$

Butterworth Filter – Impulse Invariance

$$0.89125 \le |H(e^{j\omega})| \le 1, \quad 0 \le |\omega| \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.17738, \quad 0.3\pi \le |\omega| \le \pi$$

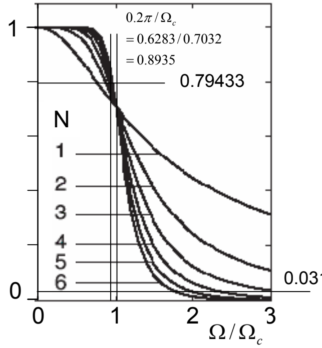
What is Ω_c? Value of N?

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \quad \text{is monotonic}$$

$$|H_c(j\Omega)|^2 = \frac{1}{1+(0.2\pi/\Omega_c)^{2N}} = (0.89125)^2 = 0.79433$$

$$|H_c(j\Omega)|^2 = \frac{1}{1+(0.3\pi/\Omega_c)^{2N}} = (0.17738)^2 = 0.03146$$

$$(0.2\pi/\Omega_c)^{2N} = 1.25893 - 1 = 0.25893$$
$$\Rightarrow \quad 2N\{\log(0.2\pi) - \log\Omega_c\} = -0.58683 \quad \Rightarrow \quad 2N\{-0.20182 - \log\Omega_c\} = -0.58683$$
$$(0.3\pi/\Omega_c)^{2N} = 31.78269 - 1 = 30.78269$$
$$\Rightarrow \quad 2N\{\log(0.3\pi) - \log\Omega_c\} = 1.50219 \quad \Rightarrow \quad 2N\{-0.02573 - \log\Omega_c\} = 1.50219$$

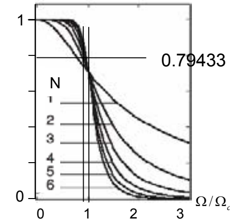


$$N \approx 5.9 \quad \Rightarrow \quad N = 6 \qquad -0.20182 - \log\Omega_c = -0.58683/12 \quad \Rightarrow \quad \Omega_c = 0.7032$$

Butterworth Filter – Impulse Invariance

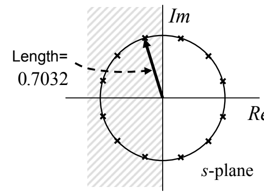
$$0.89125 \le |H(e^{j\omega})| \le 1, \quad 0 \le |\omega| \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.17738, \quad 0.3\pi \le |\omega| \le \pi$$

$$H_c(s)H_c(-s) = \frac{1}{1+(s/j\Omega_c)^{12}} \qquad N = 6 \qquad \Omega_c = 0.7032$$



$$s_k = 0.7032 \bullet \exp\left\{\frac{j\pi}{12}(2k+5)\right\} \qquad k = 0, 1, 2, \cdots, 11$$

$$H_c(s) = \frac{0.12093}{(s^2+0.3640s+0.4945)(s^2+0.9945s+0.4945)(s^2+1.3585s+0.4945)} = \sum_{k=1}^{N=6} \frac{A_k}{s-s_k}$$



$$h_c(t) = \begin{cases} \sum_{k=1}^{N=6} A_k e^{s_k t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$$h[n] = T_d h_c(nT_d)$$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

$N \approx 5.9 \Rightarrow N \approx 5.9 \Rightarrow N = 6$

Butterworth Filter- Bilinear Transform

$$0.89125 \le |H(e^{j\omega})| \le 1, \quad 0 \le |\omega| \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.17738, \quad 0.3\pi \le |\omega| \le \pi$$

$$0.89125 \le |H_c(j\Omega)| \le 1, \quad 0 \le \Omega \le \frac{2}{T_d}\tan(\frac{0.2\pi}{2})$$

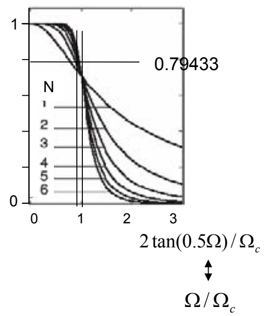
$$|H(j\Omega)| \le 0.17738, \quad \frac{2}{T_d}\tan(\frac{0.3\pi}{2}) \le \Omega \le \infty$$

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \quad \text{is monotonic}$$

$$\Rightarrow \quad 0.89125 \le |H_c(j2\tan(0.1\pi))|$$
$$|H(j2\tan(0.15\pi))| \le 0.17738$$

$$|H_c(j\Omega)|^2 = \frac{1}{1+(2\tan(0.1\pi)/\Omega_c)^{2N}} = (0.89125)^2 = 0.79433$$

$$|H_c(j\Omega)|^2 = \frac{1}{1+(2\tan(0.15\pi)/\Omega_c)^{2N}} = (0.17738)^2 = 0.03146$$



Same function approximation except on a warped scale

Bilinear Transformation of Butterworth Filter

Assume $T_d = 1$. Since a Butterworth filter has a monotonic magnitude response,

we require $\quad 0.89125 \le |H_c(j\frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right))| = |H_c(j2\tan(0.1\pi))|$

and $\quad |H_c(j2\tan(0.15\pi))| \le 0.17783$

$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}$$

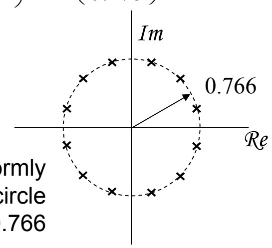
$$1 + \left(\frac{2\tan(0.1\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89}\right)^2 \quad \text{and}$$

$$1 + \left(\frac{2\tan(0.15\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.178}\right)^2$$

$$N = \frac{\log\left[\left((0.178)^{-2}-1\right)/\left((0.89)^{-2}-1\right)\right]}{2\log\left[\tan(0.15\pi)/\tan(0.1\pi)\right]} = 5.305,$$

therefore, $N = 6$.

With $N = 6$, $\quad \Omega_c = 0.766$

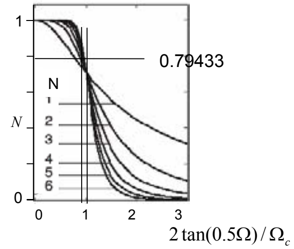


12 poles uniformly distributed on a circle of radius 0.766

Butterworth Filter- Bilinear Transform

$$|H_c(j\Omega)|^2 = \frac{1}{1+(2\tan(0.1\pi)/\Omega_c)^{2N}} = (0.89125)^2 = 0.79433$$

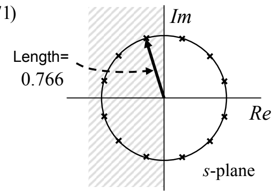
$$|H_c(j\Omega)|^2 = \frac{1}{1+(2\tan(0.15\pi)/\Omega_c)^{2N}} = (0.17738)^2 = 0.03146$$



$$N = 6 \qquad \Omega_c = 0.766$$

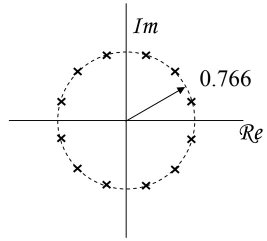
$$H_c(s) = \frac{0.20238}{(s^2+0.3996s+0.5871)(s^2+1.0836s+0.5871)(s^2+1.4802s+0.5871)}$$

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad T_d = 1$$



$$H(z) = \frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})(1-0.9044z^{-1}+0.2155z^{-2})}$$

Bilinear Transformation of Butterworth Filter



$$(s+0.198+j(0.74))(s+0.198-j(0.74))$$
$$=(s+0.198)^2+(0.74)^2 = s^2+0.396s+0.587$$
$$(s+0.541+j(0.541))(s+0.541-j(0.541))$$
$$= s^2+0.1082s+0.587$$
$$(s+0.74+j(0.198))(s+0.74-j(0.198))$$
$$= s^2+1.48s+0.587$$

Pole pair to keep:

$-0.198 \pm j(0.74)$

$-0.541 \pm j(0.541)$

$-0.74 \pm j(0.198)$

$$H_c(s) = \frac{A}{(s^2+0.396s+0.587)(s^2+1.082s+0.587)(s^2+1.48s+0.587)}$$

$A = 0.2023$ such that $H_c(0) \approx 1$

$$z = \frac{1+(s/2)}{1-(s/2)} \quad \& \quad s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{assuming } T_d = 1$$

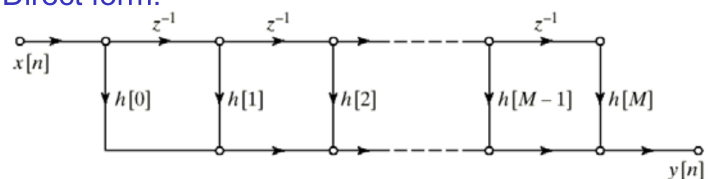
$$H(z) = \frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})(1-0.9044z^{-1}+0.2155z^{-2})}$$

FIR Filter Design by Windowing

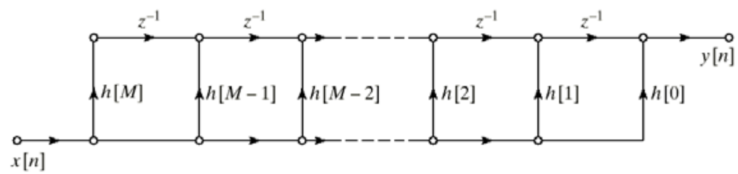
FIR Filter Structures

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] \qquad H(z) = \sum_{k=0}^{M} h[k]z^{-n}$$

- Direct form:

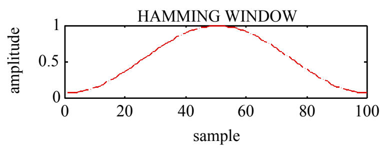


- Transposed form



Window Method for Design of FIR Filters

- *Step 1*. Compute analytically the ideal impulse response, usually by inverse Fourier transformation of an ideal frequency response
  - example: ideal low pass filter, cutoff f_0, unity gain, has ideal impulse response of infinite duration: $$h_{\text{ideal}}[n] = \frac{\sin(2\pi f_0 n)}{\pi n}$$
- Step 2. Obtain a finite-duration approximation by windowing the ideal response, *i.e.* multiplying by a tapered, finite-length function
$$h[n] = h_{\text{ideal}}[n] \bullet w[n]$$
- example: Hamming window:



$$w[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \le n \le N-1$$

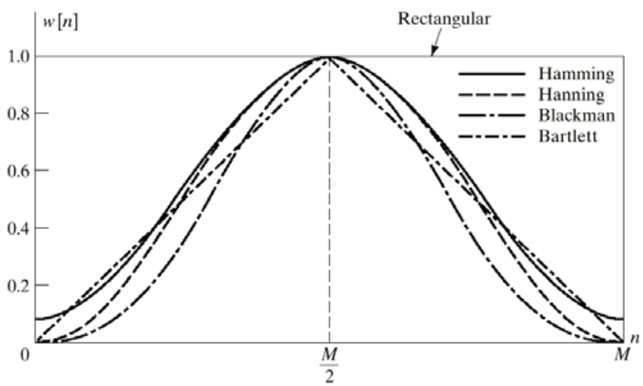
  - the window both limits the impulse response duration and controls the frequency response sidelobes
  - for a given sidelobe level, sharpness of cutoff increases with N

Common Windows

- Rectangular
$$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$
- Bartlett (triangular)
$$w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2 \\ 2 - 2n/M, & M/2 < n \le M \\ 0, & \text{otherwise} \end{cases}$$
- Hanning
$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$
- Hamming
$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Common Windows (cont'd)

- Blackman
$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$



Effect of the Window Method on Frequency Response

$$h[n] = w[n]h_d[n] \quad \Leftrightarrow \quad H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

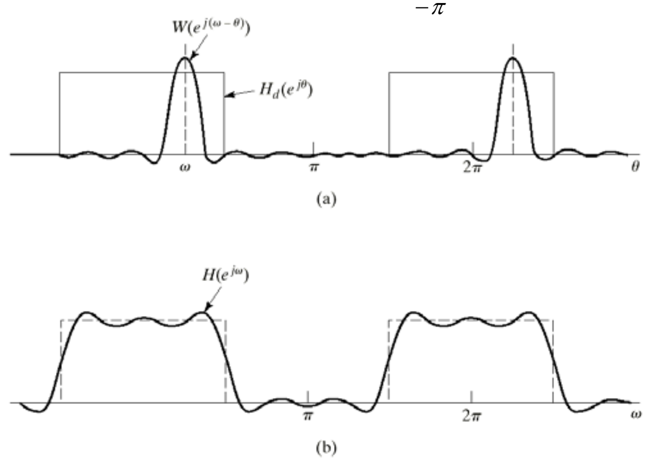
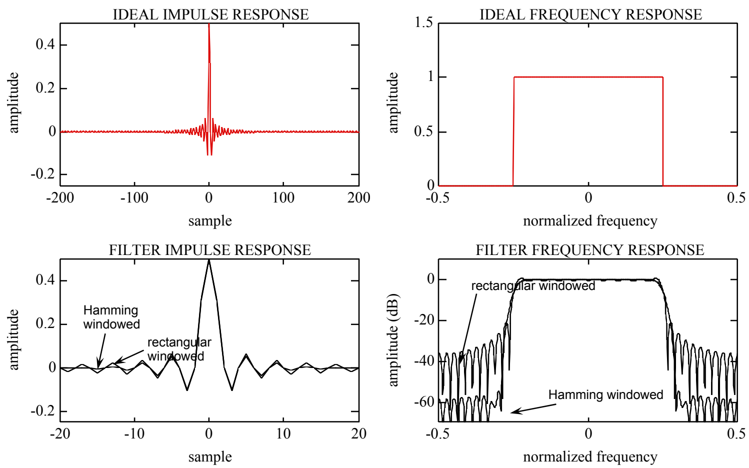


Illustration of Window Design of Half-Band Lowpass Filter



Linear Phase in Window Design - I

- Choose a symmetric causal window such that

$$w[M-n]=w[n] \quad \Leftrightarrow \quad W(e^{j\omega})=W_e(e^{j\omega})e^{-j\omega M/2}$$

$$w[n]=0, \quad \text{for } n<0 \text{ and } n>M \text{ and}$$

and either a symmetric ideal impulse response

$$h_d[M-n]=h_d[n] \quad \Leftrightarrow \quad H_d(e^{j\omega})=H_e(e^{j\omega})e^{-j\omega M/2}$$

or an anti-symmetric impulse response

$$h_d[M-n]=-h_d[n] \quad \Leftrightarrow \quad H_d(e^{j\omega})=jH_o(e^{j\omega})e^{-j\omega M/2}$$

- Then it follows that

$$h[n]=w[n]h_d[n]=\pm h[M-n], \quad 0\le n\le M.$$

Linear Phase in Window Design - II

$$h[n]=w[n]h_d[n] \quad \Rightarrow$$

$$H(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\omega$$

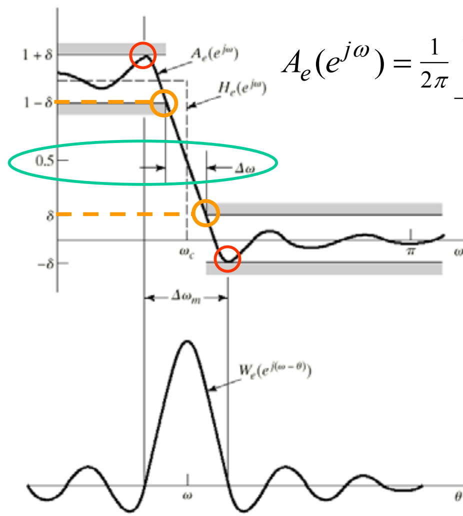
$$H(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_e(e^{j\theta})e^{-j\theta M/2}W_e(e^{j(\omega-\theta)})e^{-j(\omega-\theta)M/2}d\theta$$

$$=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}H_e(e^{j\theta})W_e(e^{j(\omega-\theta)})d\theta\right)e^{-j\omega M/2}$$

$$H(e^{j\omega})=A_e(e^{j\omega})e^{-j\omega M/2} \quad \text{where,}$$

$$A_e(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_e(e^{j\theta})W_e(e^{j(\omega-\theta)})d\omega$$

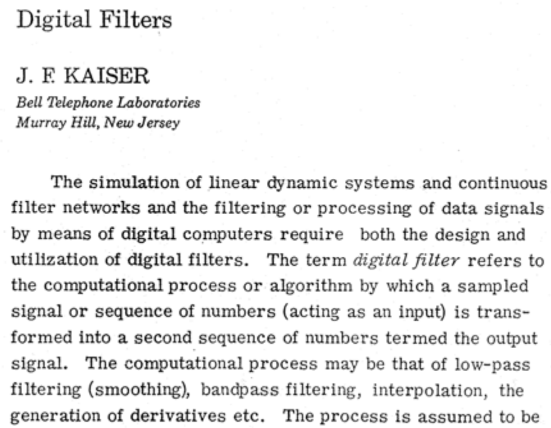
Details of the Window Approximation



$$A_e(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_e(e^{j\theta})W_e(e^{j(\omega-\theta)})d\theta$$

The phase is exactly linear with delay $M/2$

J. F. Kaiser, 1966



Digital Filters

J. F. KAISER

Bell Telephone Laboratories
Murray Hill, New Jersey

The simulation of linear dynamic systems and continuous filter networks and the filtering or processing of data signals by means of digital computers require both the design and utilization of digital filters. The term *digital filter* refers to the computational process or algorithm by which a sampled signal or sequence of numbers (acting as an input) is transformed into a second sequence of numbers termed the output signal. The computational process may be that of low-pass filtering (smoothing), bandpass filtering, interpolation, the generation of derivatives etc. The process is assumed to be

J. F. Kaiser, in *System Analysis by Digital Computer*, ed. By F. F. Kuo and J. F. Kaiser, John Wiley & Sons, 1966.

Kaiser Window Design Method

$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

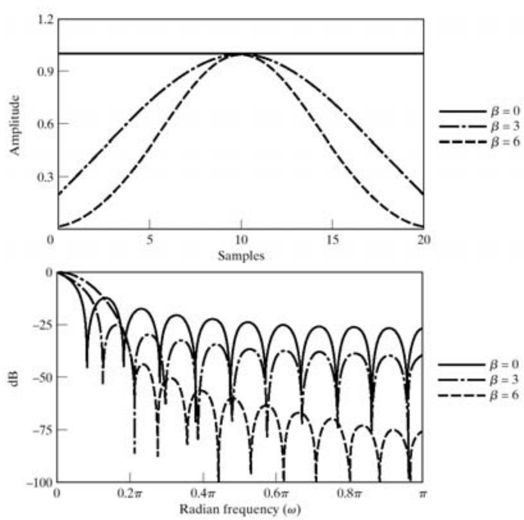
$$\alpha = M/2$$

$$\Delta\omega = \omega_s - \omega_p \quad \text{and} \quad A = -20\log_{10}\delta$$

$$M = \frac{A-8}{2.285\Delta\omega} \quad \Rightarrow \quad \text{required to meet specs}$$

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 < A < 50 \\ 0.0 & A < 21 \end{cases}$$

Kaiser Windows



Lowpass Filter Design Example

- Ideal filter:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega M/2} & |\omega| < \omega_c, \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_d[n] = \frac{\sin\omega_c(n-M/2)}{\pi(n-M/2)}$$

- Specifications:

$$\omega_p = 0.4\pi,\ \omega_s = 0.6\pi,\ \delta_1 = 0.01,\ \delta_2 = 0.001$$

$$\omega_c = \frac{\omega_p+\omega_s}{2} \quad \text{since transition is symmetric}$$

$$A = 20\log_{10}(.001) = 60 \quad \text{since error is symmetric}$$

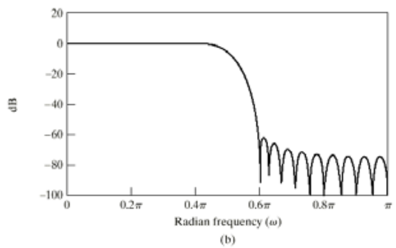
Kaiser Window Design Example

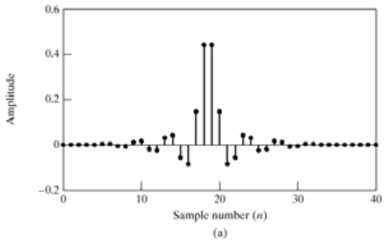
$$\omega_p = 0.4\pi,\ \omega_s = 0.6\pi$$

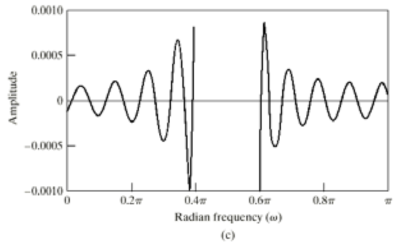
$$\Delta\omega = 0.2\pi,$$

$$A = -20\log_{10}(.001) = 60$$

$$M = \left\lceil \frac{52}{2.285(.2\pi)} \right\rceil = 37$$







Kaiser Window Design in MATLAB

```
» [N,Wc,BETA,TYPE] = KAISERORD([.4,.6],[1,0],[.01,.001],2)
N =
   37
Wn =
   0.5000
BETA =
   5.6533
TYPE =
   ''

»hd=FIR1(N,Wc,KAISER(N+1,BETA),'noscale')
```

NOTE: N is the *order* of the FIR filter in the function FIR1, but the KAISER window function wants the *length* of the window, which is the *length* of the resulting impulse response.

Analog Differentiator

$$x_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(\Omega)\, e^{j\Omega t}\, d\Omega$$

$$x_c'(t) = \frac{d}{dt}x_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\Omega X_c(\Omega)\, e^{j\Omega t}\, d\Omega$$

$$\Rightarrow F\left\{x_c'(t)\right\} = j\Omega X_c(\Omega)$$

- This suggests that the filter with frequency response

$$H(j\Omega) = j\Omega$$

acts as a *differentiator* for continuous-time signals

Digital "Differentiator"

- Suppose that we want to design an FIR filter such that

$$H_{\text{eff}}(j\Omega) = j\Omega, \quad |\Omega| < \frac{\pi}{T}$$

- The required digital filter must approximate

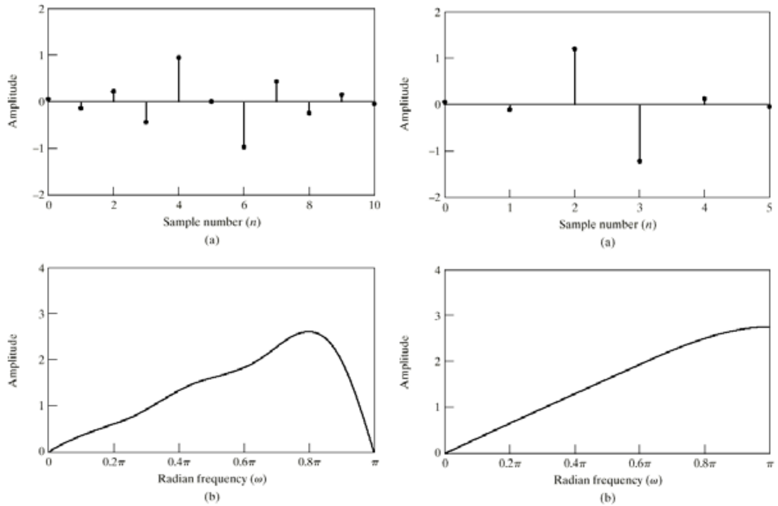
$$H(e^{j\omega}) = \frac{j\omega}{T}e^{-j\omega M/2}, \quad |\omega| < \pi$$

- The desired impulse response is

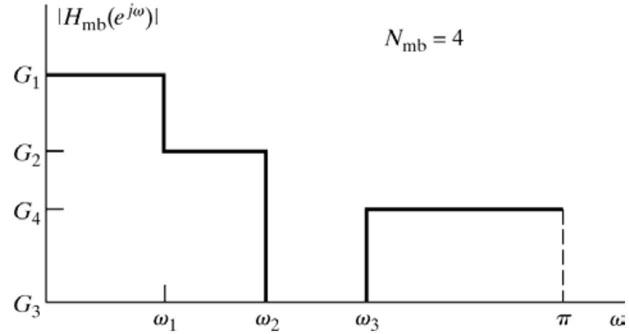
$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{j\omega}{T}\right)e^{-j\omega M/2}e^{j\omega n}\,d\omega$$

$$h_d[n] = \frac{\cos[\pi(n-M/2)]}{(n-M/2)T} - \frac{\sin[\pi(n-M/2)]}{\pi(n-M/2)T}, \quad -\infty < n < \infty$$

Kaiser Window Differentiators



General Frequency Selective Filter



$$h_{mb}[n] = \sum_{k=1}^{N_{mb}} \left(G_k - G_{k+1}\right) \frac{\sin \omega_k (n - M/2)}{\pi (n - M/2)}$$