ECE 4270
Fundamentals of Digital Signal Processing

Lecture 18: Design of Equiripple FIR Filters, Median Filter

School of Electrical and Computer Engineering
Georgia Institute of Technology
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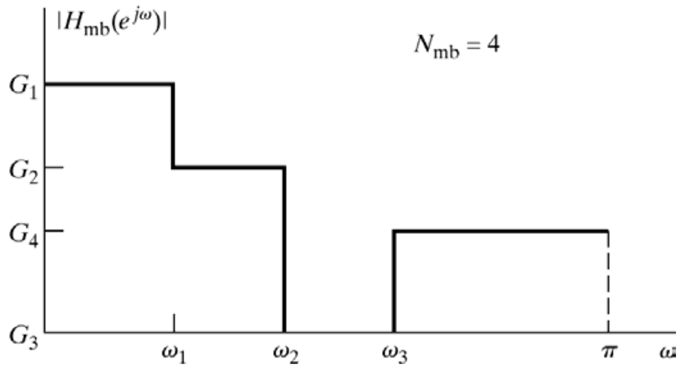
A Few More Window Design FIR Filter Points …

General Frequency Selective Filter



$$h_{mb}[n] = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin \omega_k (n - M/2)}{\pi (n - M/2)}$$

Analog Differentiator

$$x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(\Omega)\, e^{j\Omega t} d\Omega$$

$$x_c'(t) = \frac{d}{dt} x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\Omega X_c(\Omega)\, e^{j\Omega t} d\Omega$$

$$\Rightarrow F\left\{x_c'(t)\right\} = j\Omega X_c(\Omega)$$

- This suggests that the filter with frequency response

$$H(j\Omega) = j\Omega$$

acts as a *differentiator* for continuous-time signals

Digital "Differentiator"

- Suppose that we want to design an FIR filter such that

$$H_{\text{eff}}(j\Omega) = j\Omega, \quad |\Omega| < \frac{\pi}{T}$$

- The required digital filter must approximate

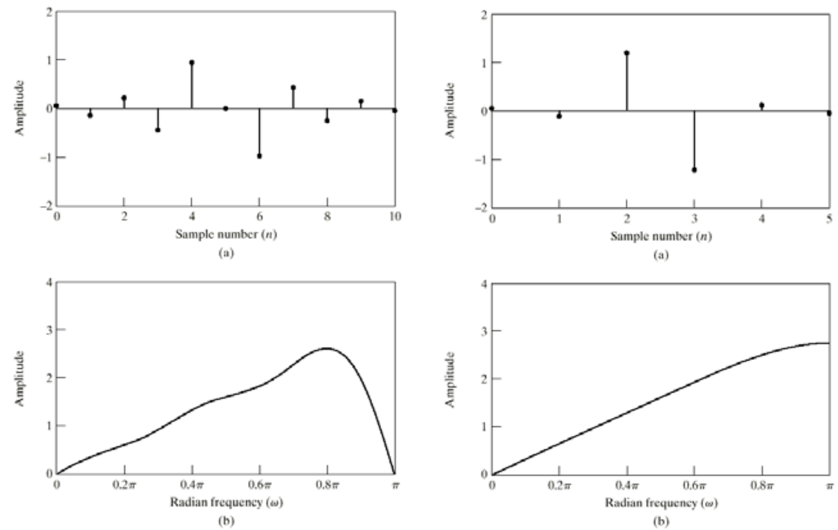
$$H(e^{j\omega}) = \frac{j\omega}{T} e^{-j\omega M/2}, \quad |\omega| < \pi$$

- The desired impulse response is

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{j\omega}{T}\right) e^{-j\omega M/2} e^{j\omega n} d\omega$$

$$h_d[n] = \frac{\cos[\pi(n - M/2)]}{(n - M/2)T} - \frac{\sin[\pi(n - M/2)]}{\pi(n - M/2)T}, \quad -\infty < n < \infty$$

Kaiser Window Differentiators



Equiripple Design of FIR Filters

Parks and McClellan, 1972

Chebyshev Approximation for Nonrecursive Digital Filters with Linear Phase

THOMAS W. PARKS, MEMBER, IEEE, AND JAMES H. McCLELLAN, STUDENT MEMBER, IEEE

Abstract—An efficient procedure for the design of finite-length impulse response filters with linear phase is presented. The algorithm obtains the optimum Chebyshev approximation on separate intervals corresponding to passbands and/or stopbands, and is capable of designing very long filters. This approach allows the exact specification of arbitrary band-edge frequencies as opposed to previous algorithms which could not directly control pass- and stopband locations and could only obtain $(N-1)/2$ different band-edge locations for a length N low-pass filter, for fixed δ_1 and δ_2.

As an aid in practical application of the algorithm, several graphs are included to show relations among the parameters of filter length, transition width, band-edge frequencies, passband ripple, and stopband attenuation.

capable of designing longer filters. The algorithms in [7], [8] result in exactly the same filter and will be called an extraripple design in this paper.

The detailed description of the new procedure described here is in terms of low-pass filters. Modifications for the general bandpass case are included. Linear-phase digital filters of length $2n+1$ have a transfer function

$$G(Z) = \sum_{k=0}^{2n} h_k Z^{-k} \qquad (1)$$

T. W. Parks and J. H. McClellan, *IEEE Trans. Circuit Theory*, CT-19, pp. 189-194, March, 1972.

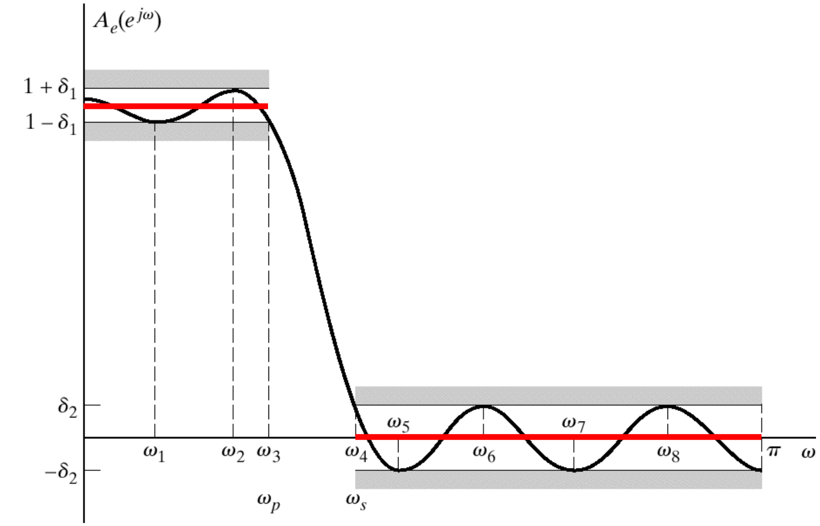
The Parks McClellan Algorithm

- Uses the Remez exchange algorithm to iteratively find the impulse response that minimizes the maximum approximation error over a set of closed intervals in the frequency-domain.

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - H(e^{j\omega})]$$

- Leads to equiripple approximations that are optimum in sense of smallest approximation error for a given transition width.

Optimum FIR Filter



Aside: Polynomials in $\cos\omega$

- Every term of the form $\cos(n\omega)$ can be replaced by an n^{th}-order polynomial in $\cos\omega$
- Examples (trig identities):

$$\cos 2\omega = -1 + 2\cos^2\omega$$

$$\cos 3\omega = -3\cos\omega + 4\cos^3\omega$$

$$\cos 4\omega = 1 - 8\cos^2\omega + 8\cos^4\omega$$

- This means we can write (for some set of coefficients)

$$\sum_{n=0}^{L} \alpha_n \cos\omega n = \sum_{k=0}^{L} a_k (\cos\omega)^k$$

Linear Phase Type I FIR Filter

- Zero-phase impulse response:

$$h_e[-n] = h_e[n] \quad -L \le n \le L$$

$$L = \frac{M}{2} \quad (M \text{ even}) \qquad = \frac{M-1}{2} \quad (M \text{ odd})$$

- Frequency response:

$$A_e(e^{j\omega}) = \sum_{n=-L}^{L} h_e[n] e^{-j\omega n}$$

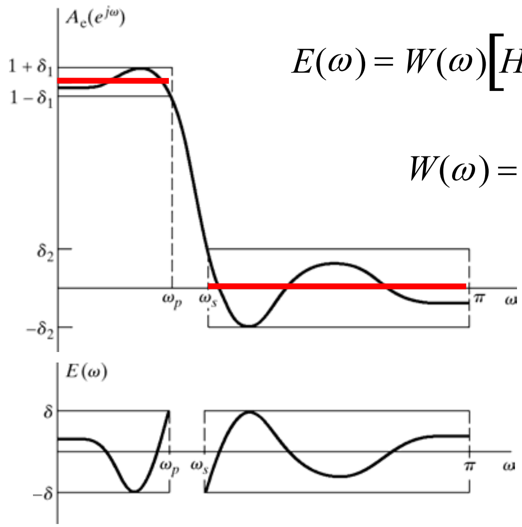
$$= h_e[0] + \sum_{n=1}^{L} \left(h_e[n] e^{-j\omega n} + h_e[-n] e^{j\omega n} \right)$$

$$= h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos\omega n = \sum_{k=0}^{L} a_k (\cos\omega)^k$$

- Causal version:

$$h[n] = h_e[n-L] \quad \Leftrightarrow \quad H(e^{j\omega}) = A_e(e^{j\omega}) e^{-j\omega L}$$

Weighted Approximation Error



$$E(\omega) = W(\omega)\left[H_d(e^{j\omega}) - A_e(e^{j\omega})\right]$$

$$W(\omega) = \begin{cases} \dfrac{\delta_2}{\delta_1} & 0 \le \omega \le \omega_p \\ 1 & \omega_s \le \omega \le \pi \end{cases}$$

The Alternation Theorem

- Weighted approximation error:

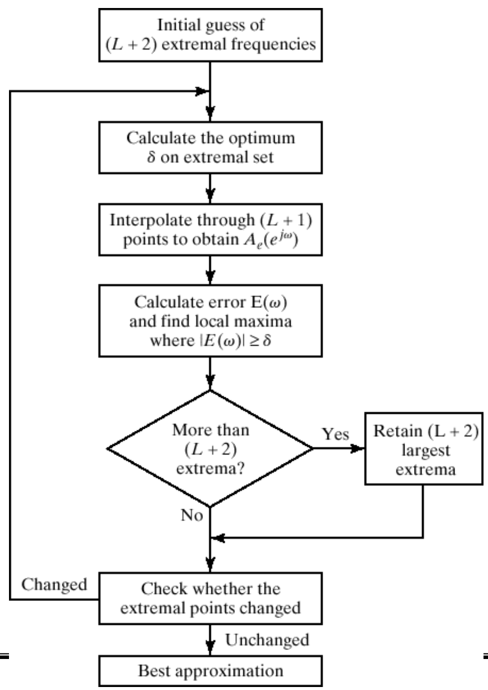
$$E(\omega) = W(\omega)\left[H_d(e^{j\omega}) - A_e(e^{j\omega})\right]$$

- Minimize the maximum error over a set of frequencies:

$$F = \{\omega: 0 \le \omega \le \omega_p \text{ and } \omega_s \le \omega \le \pi\}$$

$$\|E\| = \max_{\omega \in F}\left[|E(\omega)|\right] \qquad \delta = \min_{h_e[n]}\{\|E\|\}$$

- ***The optimum approximation alternates between +δ and -δ at least $L+2$ times in F. The maximum number of alternations is $L+3$.***



Design Formula

- Kaiser obtained the following design formula *for equiripple filters* by curve fitting many examples:

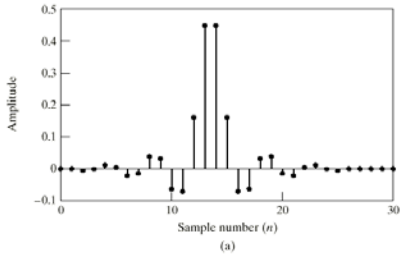
$$M = \frac{-10\log_{10}(\delta_1\delta_2) - 13}{2.324\Delta\omega}$$

- MATLAB example:
  - » [M,Fo,Mo,W] = remezord([.4,.6], [1 0], [0.01 0.001], 2);
  - » [h,delta]=remez(M,Fo,Mo,W);

Parks McClellan Lowpass Design

$$M = 27$$

$$\Rightarrow L = 13 \Rightarrow 15 \text{ or } 16 \text{ alternations}$$



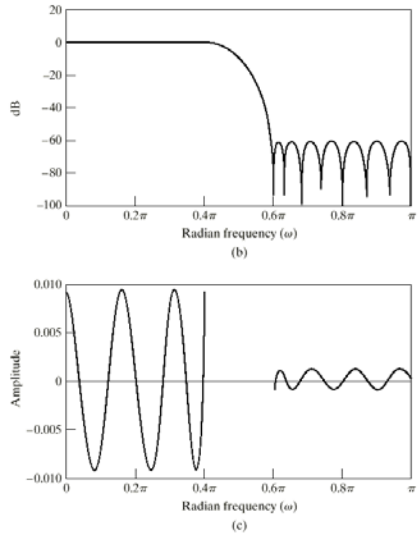
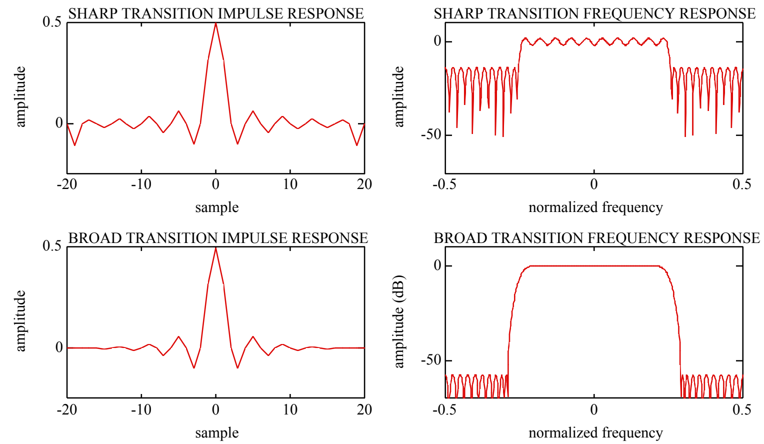


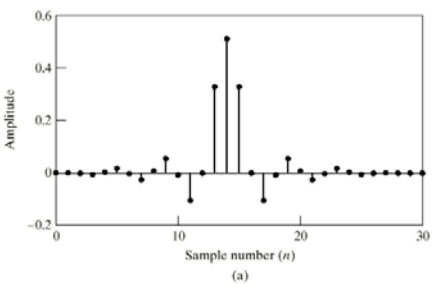
Illustration of Parks-McClellan Design of Half-Band Lowpass Filter

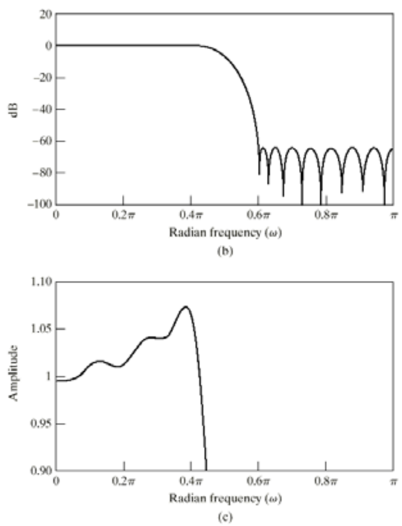
- Passband and stopband ripples can be weighted to be unequal
- Equiripple design not necessarily good in all applications
- Algorithm widely available



Parks McClellan D/A Compensated Filter

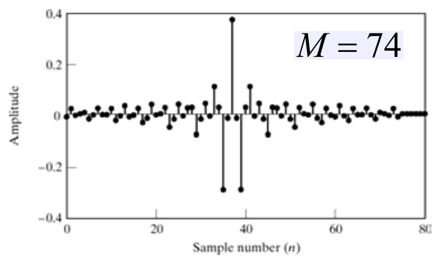
$$M = 28$$



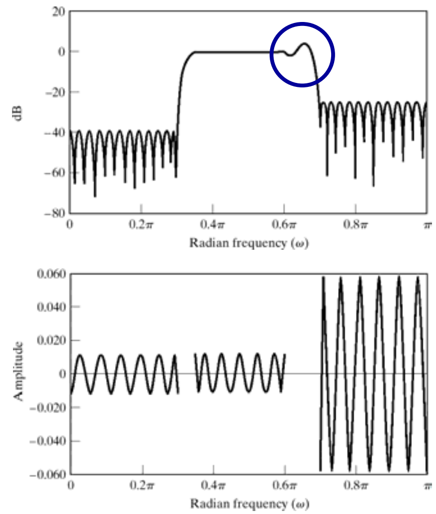


Parks-McClellan Bandpass Filter

$$H_d(e^{j\omega}) = \begin{cases} 0, & 0 \le \omega \le .3\pi \\ 1, & .35\pi \le \omega \le .6\pi \\ 0, & .7\pi \le \omega \le \pi \end{cases}$$



$$W(\omega) = \begin{cases} 1, & 0 \le \omega \le .3\pi \\ 1, & .35\pi \le \omega \le .6\pi \\ .2, & .7\pi \le \omega \le \pi \end{cases}$$



Comparison of Filter Structures

- Complexity is proportional to amount of computation and storage plus program storage and computational cycles.
- FIR direct form - $(M+1)$ coefficients
  - $(M+1)$ multiplications, M additions per output sample
    - If linear phase design, number of multiplications is halved
  - $(M+1)$ coefficients, M delays (registers)
- IIR cascade form - N_s second-order sections.
  - $5N_s$ multiplications, $5N_s$ additions per output sample
  - $5N_s$ coefficients, $5N_s$ delays

Lowpass Filter Implementations

- Specifications of lowpass filter:

$$1/T = 2000 \text{ Hz}$$

$$0.99 \le |H(e^{j\Omega T})| \le 1.01 \qquad 0 \le |\Omega| \le 2\pi(400)$$

$$|H(e^{j\Omega T})| \le 0.001 \qquad 2\pi(600) \le |\Omega| \le 2\pi(1000)$$

- These specs met or exceeded by the following filters:
  - Butterworth - 14th-order
  - Chebyshev - 8th-order
  - Elliptic - 6th-order
  - Kaiser window - 37 sample impulse response
  - Parks-McClellan - 27 sample impulse response

Comparison of Lowpass Filters

Approx. Method	Order M or N	Total Mults.	Total Adds	Total Storage	TMS320 Cycles
Butter	14	35	28	49	109
Cheby	8	20	16	28	64
elliptic	6	~~18~~ → *19*	12	21	49
Kaiser	37	~~38~~ → *14*	37	74	52
P-Mc	27	28	27	54	42

(circled) = "best"

Multiplies reduced if a linear phase design!

Summary Comparison of FIR and IIR Filters

Filter Class	Computation Required	Amplitude Control	Phase Control	Stability	Implementation Methods	Design Methods
IIR	❑ Usually minimum for given amplitude spec	❑ Best for given filter order ❑ Limited to LPF, BPF, *etc.*	❑ Usually not available ❑ Highly nonlinear phase at bandedges	❑ Standard designs guarantee stability ❑ Quantization can make unstable	❑ Difference equation	❑ Transformations of analog prototypes ❑ Computer-aided design ❑ Usually only amplitude control
FIR	❑ Greater than IIR ❑ Can be minimized through symmetries	❑ Can handle arbitrary response shapes	❑ Exactly linear phase possible	❑ Always stable	❑ Direct convolution ❑ FFT-based "fast convolution"	❑ Window method ❑ Computer-aided design

Nonlinear Filter Example: The Median Filter

- Nonlinear filters useful for removing outliers, impulse noise
- “Order statistics” filters (*e.g.*, median) are most common by far
- Can preserve sharp features better than linear filters
- **Nonlinear operation spreads spectrum**, unlike linear
- Often characterized by "root" signals