ECE 4270
Fundamentals of Digital Signal Processing

Lecture 19: The Discrete Fourier Transform (DFT)

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Georgia Institute of Technology
Summer, 2004

The Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad k = 0,\dots,N-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad n = 0,\dots,N-1$$

where $W_N \equiv e^{-j2\pi/N}$

- Exact representation of finite-length or periodic ($x[n+N]=x[n]$) sequences.
- $X[k]$ and $x[n]$ can be computed efficiently by the *fast Fourier transform* (FFT)
  - Gauss knew about it, Cooley and Tukey rediscovered it at just the right time

Why Another Fourier Transform?

- DTFT: $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$
- The DTFT is a very fine transform, *but* …
- We can only compute a function at a finite set of values of the independent variable – in the case of DTFT, only at a finite set of frequencies
- So we need a discrete and finite frequency variable!
- What if this set of frequencies conform to a certain regularity - the most natural being N such frequencies uniformly distributed over 2π?

A Simple (but important) Example

- Let $P[k]=1$, for $k=0,1,2,\dots,N$-1. Then

$$p[n] = \frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi/N)kn} = \frac{1}{N}\frac{1-e^{j(2\pi/N)nN}}{1-e^{j(2\pi/N)n}}$$

$$p[n] = \begin{cases} 1 & n=0,\pm N,\pm 2N,\dots \\ 0 & \text{otherwise} \end{cases} = \sum_{r=-\infty}^{\infty} \delta[n+rN]$$

- Generalization: DFTs (and inverse DFTs) are inherently periodic with period N.

$$p[n+N] = \frac{1}{N}\sum_{k=0}^{N-1} P[k] e^{j(2\pi/N)k(n+N)} = p[n]$$

The Same Example in "*W*" Notation

- Let $P[k]=1$, for $k=0,1,2,\ldots,N-1$. Then

$$p[n]=\frac{1}{N}\sum_{k=0}^{N-1}W_N^{kn}=\frac{1}{N}\cdot\frac{1-W_N^{nN}}{1-W_N^{n}}$$

$$p[n]=\begin{cases}1 & n=0,\pm N,\pm 2N,\ldots\\ 0 & \text{otherwise}\end{cases}=\sum_{r=-\infty}^{\infty}\delta[n+rN]$$

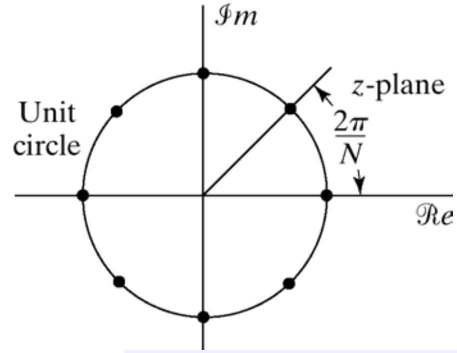
- DFTs (and inverse DFTs) are inherently periodic with period N.

$$p[n+N]=\frac{1}{N}\sum_{k=0}^{N-1}P[k]W_N^{k(n+N)}=p[n]$$

The DFT as a Sampled z-Transform

$$X(z)=\sum_{n=0}^{N-1}x[n]z^{-n}$$

$$z=e^{j(2\pi/N)kn},\quad k=0,1,2,\ldots,N-1$$



$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j(2\pi/N)kn}$$

... assuming $x[n]$ is a finite length sequence

$$=X(z)\Big|_{z=e^{j(2\pi/N)k}},\quad k=0,1,2,\ldots,N-1$$

The DFT as a Sampled DTFT

- The DTFT of an N-point sequence is

$$X(e^{j\omega})=\sum_{n=0}^{N-1}x[n]e^{-j\omega n}$$

- Sample the DTFT at $\omega_k=(2\pi/N)k,\ k=0,1,\ldots,N-1.$
- The result is identical to the DFT

$$X(e^{j\omega})\Big|_{\omega=2\pi k/N}=\sum_{n=0}^{N-1}x[n]e^{-j(2\pi/N)kn}=X[k]$$

- If we compute the inverse DFT, we obtain

$$\tilde{x}[n]=\frac{1}{N}\sum_{n=0}^{N-1}X(e^{j(2\pi/N)k})\,e^{j(2\pi/N)kn}=\sum_{r=-\infty}^{\infty}x[n+rN]$$

If We're Sampling in Frequency, Shouldn't There Be Replication in the Time Domain?

DFT Sampling Theorem

- If we sample the DTFT of $x[n]$ at N equally spaced frequencies, the corresponding periodic sequence (through the inverse DFT) is the time-domain aliased sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

- Then if $x[n]=0$ for $n<0$ and for $n>N$, the copies of $x[n]$ do not overlap so we can write

$$\tilde{x}[n] = x[((n))_N] = x[n \text{ modulo } N]$$

- Therefore:

$$x[n] = \begin{cases} \tilde{x}[n] & n = 0,1,\ldots,N-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof of DFT Sampling Theorem

- Consider a signal with DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Sample it to get a DFT

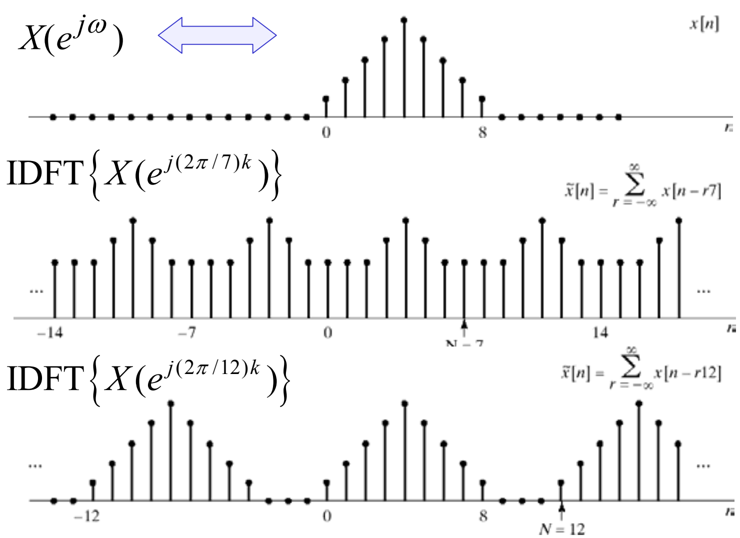
$$X[k] = X(e^{j(2\pi/N)k}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(2\pi/N)kn}$$

- Compute inverse DFT

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x[m]e^{-j(2\pi/N)km}e^{j(2\pi/N)kn}$$

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\underbrace{\frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi/N)k(n-m)}}_{\tilde{p}[n-m]} = \sum_{r=-\infty}^{\infty} x[n-rN]$$

Time-Domain Replication & Aliasing



Repetition in Time and Frequency, and Principal Intervals

- We have signals repeating themselves periodically in both time and frequency:

$$X[k] = X[k + lN], \quad l = -\infty,\ldots,\infty$$

$$\tilde{x}[n] = \tilde{x}[n + lN], \quad l = -\infty,\ldots,\infty$$

$$\left(\text{and } \tilde{x}[n] = x[n], \quad n = 0,\ldots,N-1\right)$$

- We are usually *interested* only in evaluating these signals on the *principal interval* of n or $k = 0,\ldots,N$-1

Modulo Notation

- It is convenient sometimes to use modulo evaluation of indices to represent the periodic representation of a sequence; if

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

- then we can write

$$\tilde{x}[n] = x[n \text{ modulo } N]$$

- and we denote this by

$$\tilde{x}[n] = x[n \text{ modulo } N] = x[((n))_N]$$

- Note that it follows that

$$x[((n))_N] = x[((n + lN))_N], \quad x[((-n))_N] = x[((lN - n))_N], \quad etc.$$

Example of Sampling the DTFT

- Input sequence: $x[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$
- DTFT

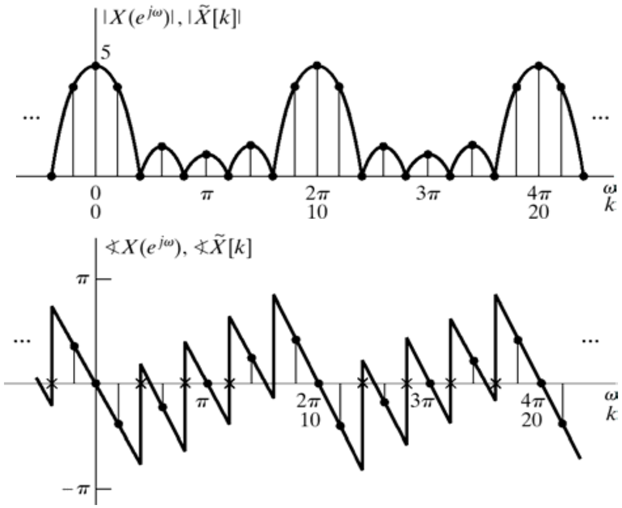
$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = \frac{1 - e^{-j\omega 5}}{1 - e^{-j\omega}} = \frac{\sin(5\omega/2)}{\sin(\omega/2)} e^{-j\omega 2}$$

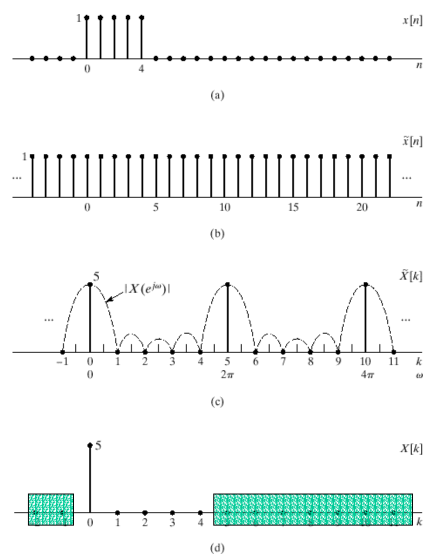
- Sampled DTFT is the DFT

$$X(e^{j(2\pi/N)k}) = \sum_{n=0}^{4} e^{-j(2\pi/N)kn} = X[k]$$

$$= \frac{\sin(5(2\pi/N)k/2)}{\sin((2\pi/N)k/2)} e^{-j(2\pi/N)k2}$$

Sampled DTFT is DFT





$x[n] = 0 \quad n < 0 \text{ and } n > M$

$M = 4$

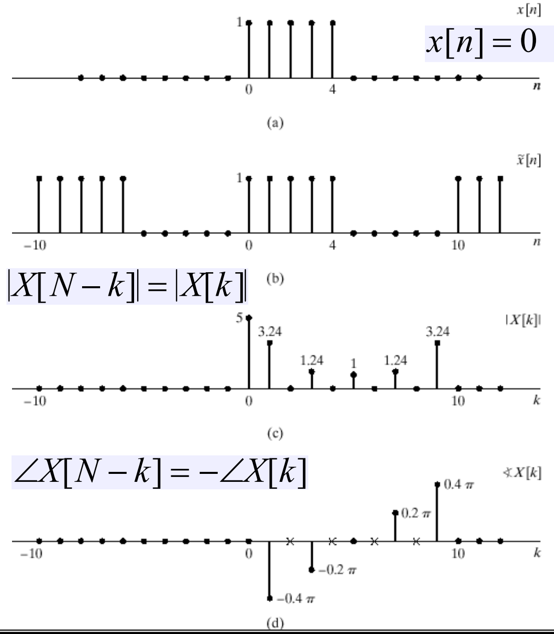
$\tilde{x}[n] = x[((n))_N]$

$N = 5$

$X[k] = X(e^{j(2\pi/N)k})$

$X[k] = X(e^{j(2\pi/N)k}),$

$k = 0, 1, \ldots, N-1$



$x[n] = 0 \quad n < 0 \text{ and } n > M$

$M = 4$

$\tilde{x}[n] = x[((n))_N]$

$N = 10$

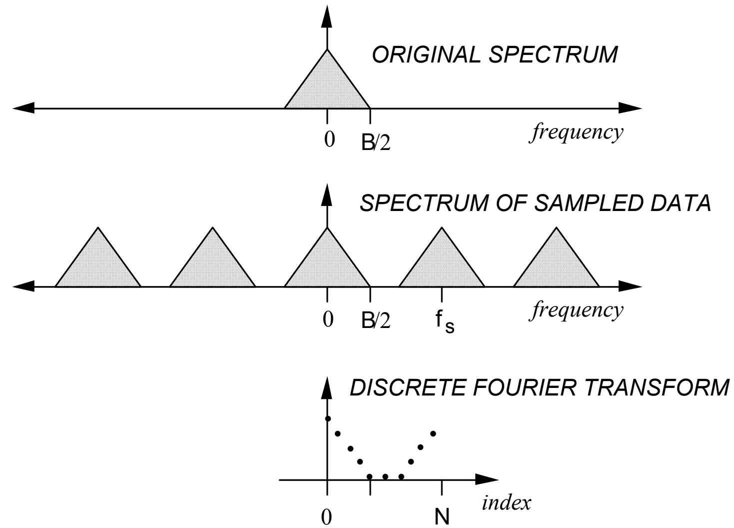
$|X[N-k]| = |X[k]|$

$X[k] = X(e^{j(2\pi/N)k}),$
$k = 0, 1, \ldots, N-1$

$\angle X[N-k] = -\angle X[k]$

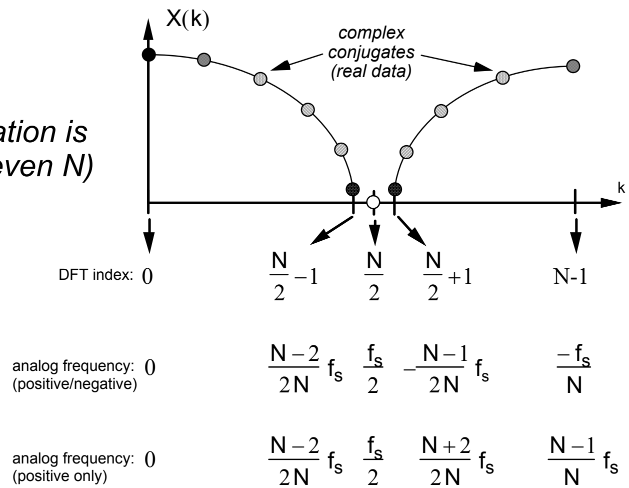
$X[N-k] = X^*[k],$
$k = 0, 1, \ldots, N-1$

Relation Between Analog and DFT Spectra



Interpretation of DFT Frequencies

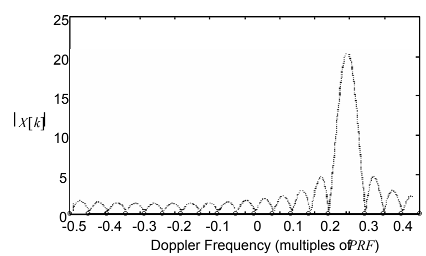
(illustration is for even N)

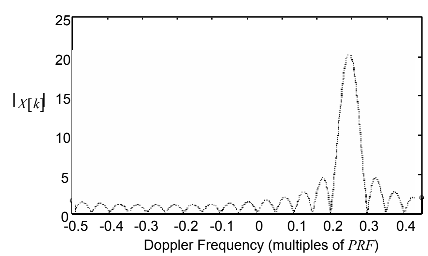


Sampling the DTFT Again

- The DFT is a sampled version of the DTFT
  - to get denser set of samples, increase N (zero padding)
  - sample points are fixed on the frequency axis
  - similar signals can have very different DFTs due to alignment

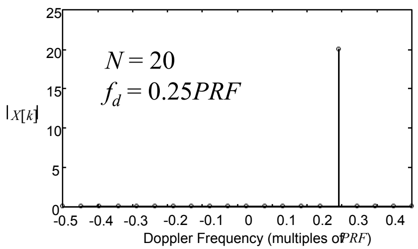
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}$$
$$= X(\omega)\big|_{\omega = 2\pi k/N}$$

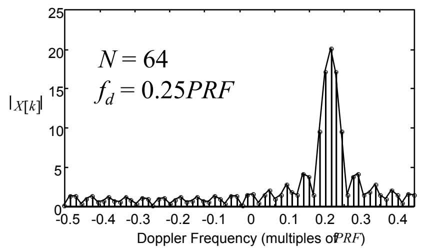




Appearance of DFT Depends on Sampling Density

- Sampling density increased by zero padding
  - underlying DTFT is not changed

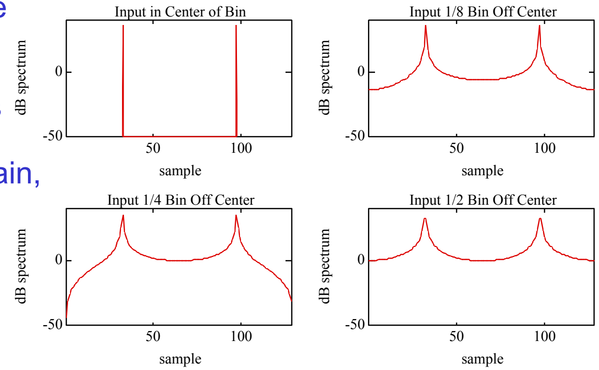




Appearance of DFT Depends on Alignment of Signal & DFT Frequencies

- DFT sample frequencies are fixed for given N
- DTFT alignment relative to DFT samples moves with signal frequency
  - sampling the same DTFT in different places gives apparent change in gain, resolution, sidelobes

N=128



Terminology

- We have defined two flavors of Fourier transforms for discrete-index signals, with a third yet to come:
  - DTFT: Discrete-Time Fourier Transform
    - continuous frequency variable
  - DFT: Discrete Fourier Transform
    - discrete frequency variable
  - FFT: Fast Fourier transform
    - *not* a different form of Fourier transform
    - just a fast *algorithm* for computing the DFT

DFT Theorems and Properties

	Finite-Length Sequence (Length N)	N-point DFT (Length N)
1.	$x[n]$	$X[k]$
2.	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km}X[k]$
6.	$W_N^{-\ell n}x[n]$	$X[((k-\ell))_N]$
7.	$\sum_{m=0}^{N-1} x_1(m)x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8.	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell)X_2[((k-\ell))_N]$

Basic Properties of the DFT

- If the inverse DFT is evaluated outside of $0 \le n \le N-1$, it repeats periodically as $\tilde{x}[n] = x[((n))_N]$.

$$\tilde{x}[n+N] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-k(n+N)} = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn} = \tilde{x}[n]$$

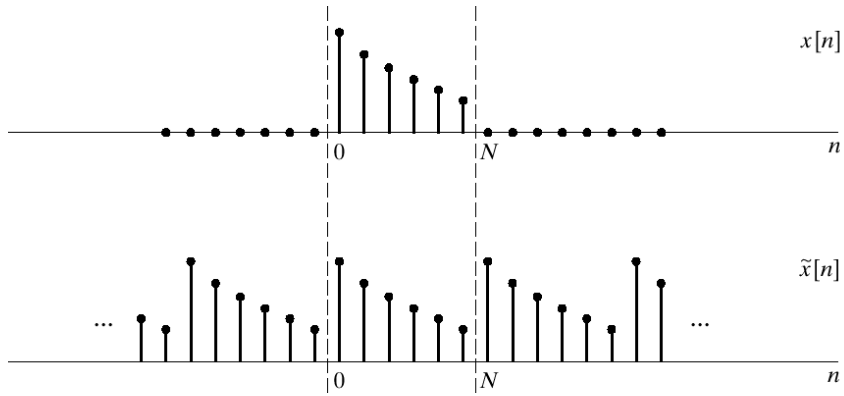
- Circular shift

$$x_1[n] = x[((n-m))_N] \quad \Leftrightarrow \quad X_1[k] = W_N^{km} X[k]$$

- Circular convolution

$$y[n] = \sum_{m=0}^{N-1} x[m] h[((n-m))_N] \quad \Leftrightarrow \quad Y[k] = X[k]H[k]$$

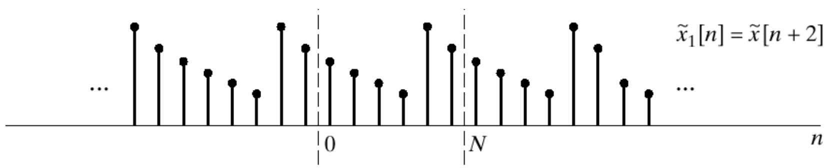
Circular Shift - I



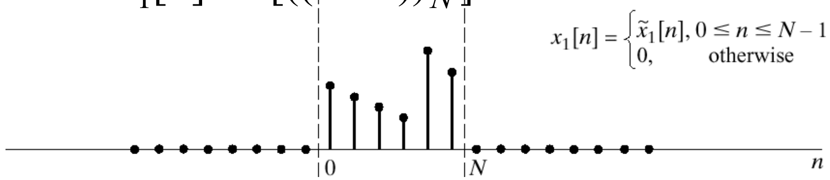
$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN] = x[((n))_N]$$

Circular Shift - II

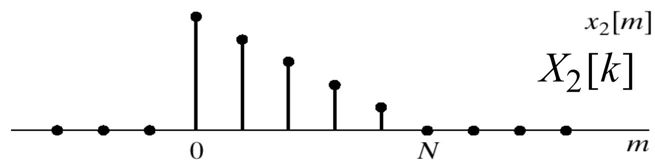
$$\tilde{x}_1[n] = \tilde{x}[n+2] = \sum_{r=-\infty}^{\infty} x[n+2+rN]$$

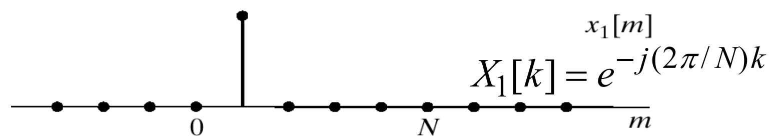


$$x_1[n] = x[((n+2))_N]$$



Circular Convolution Example





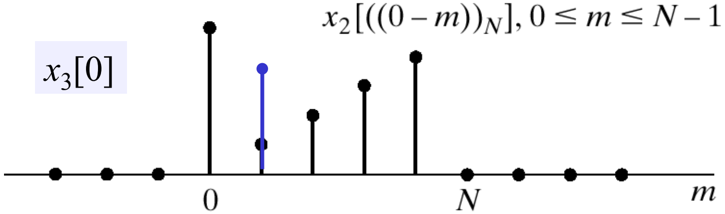
$$X_3[k] = X_1[k] X_2[k] = e^{-j(2\pi/N)k} X_2[k]$$

$$x_3[n] = \sum_{m=0}^{N-1} \delta[m-1] x_2[((n-m))_N] = x_2[((n-1))_N]$$

Circular Flipping and Shifting

$x_2[((0-m))_N], 0 \le m \le N-1$

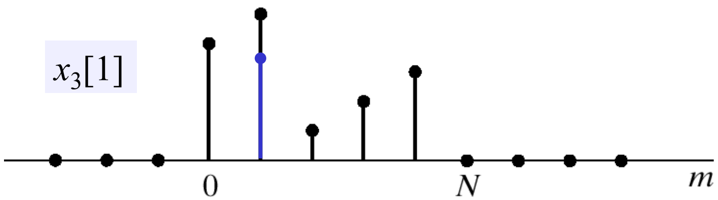
$x_3[0]$



$$x_3[n] = \sum_{m=0}^{N-1} \delta[m-1] x_2[((n-m))_N]$$

$x_2[((1-m))_N], 0 \le m \le N-1$

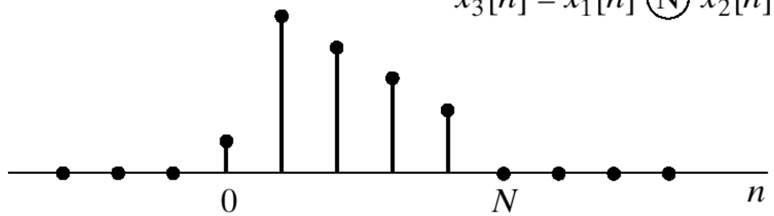
$x_3[1]$



Circular Convolution Example, concluded

$$x_3[n] = \sum_{m=0}^{N-1} \delta[m-1] x_2[((n-m))_N]$$

$x_3[n] = x_1[n] \circledN x_2[n]$



$$x_3[n] = x_2[((n-1))_N]$$