ECE 4270
Fundamentals of Digital Signal Processing

Lecture 20: Convolution Using The Discrete Fourier Transform

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The Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,\dots,N-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad n = 0,\dots,N-1$$

where $W_N \equiv e^{-j2\pi/N}$

- Exact representation of finite-length or periodic ($x[n+N]=x[n]$) sequences.
- $X[k]$ and $x[n]$ can be computed efficiently by the *fast Fourier transform* (FFT)
  – Gauss knew about it, Cooley and Tukey rediscovered it at just the right time

Why Another Fourier Transform?

- DTFT: $$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$
- The DTFT is a very fine transform, *but* …
- We can only compute a function at a finite set of values of the independent variable – in the case of DTFT, only at a finite set of frequencies
- So we need a discrete and finite frequency variable!
- What if this set of frequencies conform to a certain regularity - the most natural being N such frequencies uniformly distributed over 2π?

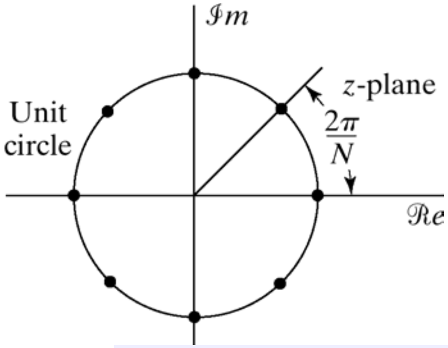
Terminology

- We have defined two flavors of Fourier transforms for discrete-index signals, with a third yet to come:
  – DTFT: Discrete-Time Fourier Transform
    • continuous frequency variable
  – DFT: Discrete Fourier Transform
    • discrete frequency variable
  – FFT: Fast Fourier transform
    • *not* a different form of Fourier transform
    • just a fast *algorithm* for computing the DFT

The DFT as a Sampled z-Transform

$$X(z) = \sum_{n=0}^{N-1} x[n] z^{-n}$$

$$z = e^{j(2\pi/N)kn},\quad k = 0,1,2,\ldots,N-1$$



$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$= X(z)\Big|_{z=e^{j(2\pi/N)k}},\quad k = 0,1,2,\ldots,N-1$$

... assuming $x[n]$ is a finite length sequence

The DFT as a Sampled DTFT

- The DTFT of an N-point sequence is

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

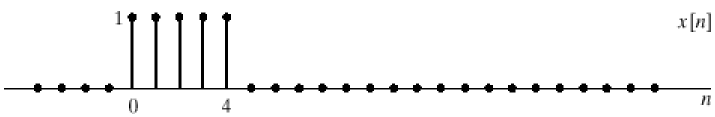
- Sample the DTFT at $\omega_k = (2\pi/N)k,\ k = 0,1,\ldots,N-1.$
- The result is identical to the DFT

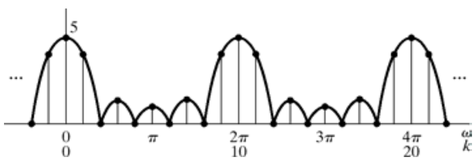
$$X(e^{j\omega})\Big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} = X[k]$$

- If we compute the inverse DFT, we obtain

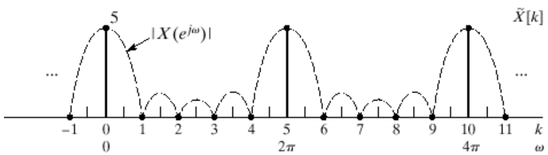
$$\tilde{x}[n] = \frac{1}{N}\sum_{n=0}^{N-1} X(e^{j(2\pi/N)k})\, e^{j(2\pi/N)kn} = \sum_{r=-\infty}^{\infty} x[n+rN]$$

Sampled DTFT is DFT





$N = 10$ point DFT



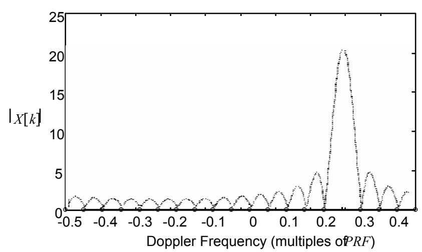
$N = 5$ point DFT

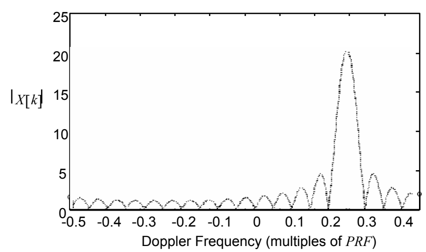
Sampling the DTFT Again

- The DFT is a sampled version of the DTFT
  - to get denser set of samples, increase N (zero padding)
  - sample points are fixed on the frequency axis
  - similar signals can have very different DFTs due to alignment

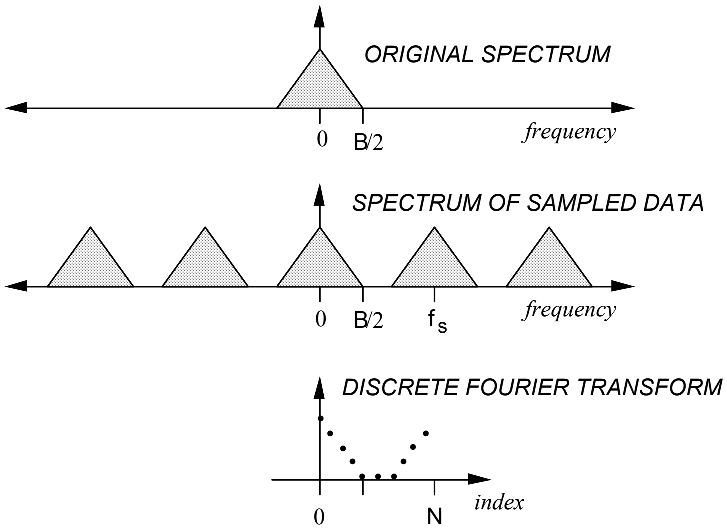
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}$$

$$= X(\omega)\Big|_{\omega=2\pi k/N}$$

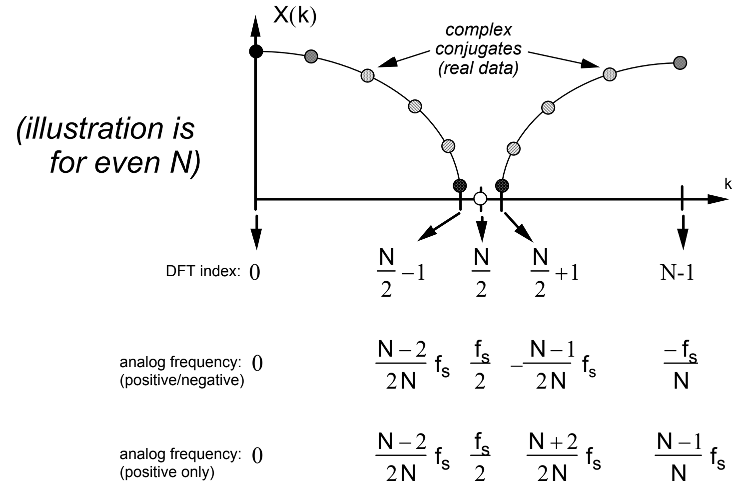




Relation Between Analog and DFT Spectra



Interpretation of DFT Frequencies



DFT Sampling Theorem

- If we sample the DTFT of $x[n]$ at N equally spaced frequencies, the corresponding periodic sequence (through the inverse DFT) is the time-domain aliased sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$

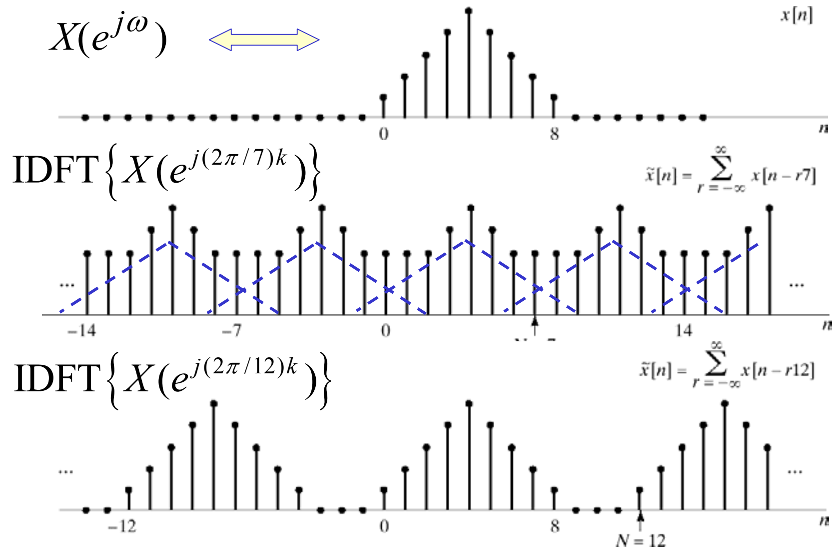
- Then if $x[n]=0$ for $n<0$ and for $n>N$, the copies of $x[n]$ do not overlap so we can write

$$\tilde{x}[n] = x[((n))_N] = x[n \text{ modulo } N]$$

- Therefore:

$$x[n] = \begin{cases} \tilde{x}[n] & n = 0,1,\ldots,N-1 \\ 0 & \text{otherwise} \end{cases}$$

Time-Domain Replication & Aliasing



Modulo Notation

- It is convenient sometimes to use modulo evaluation of indices to represent the periodic representation of a sequence; if

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

- then we can write

$$\tilde{x}[n] = x[n \text{ modulo } N]$$

- and we denote this by

$$\tilde{x}[n] = x[n \text{ modulo } N] = x[((n))_N]$$

- Note that it follows that

$$x[((n))_N] = x[((n+lN))_N], \quad x[((-n))_N] = x[((lN-n))_N], \quad etc.$$

Basic Properties of the DFT

- If the inverse DFT is evaluated outside of $0 \le n \le N-1$, it repeats periodically as $\tilde{x}[n] = x[((n))_N]$.

$$\tilde{x}[n+N] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-k(n+N)} = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn} = \tilde{x}[n]$$

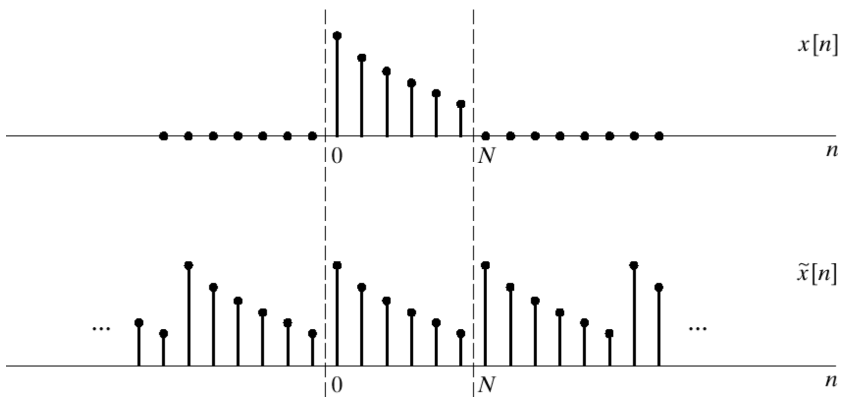
- Circular shift

$$x_1[n] = x[((n-m))_N] \quad \Leftrightarrow \quad X_1[k] = W_N^{km} X[k]$$

- Circular convolution

$$y[n] = \sum_{m=0}^{N-1} x[m] h[((n-m))_N] \quad \Leftrightarrow \quad Y[k] = X[k]H[k]$$

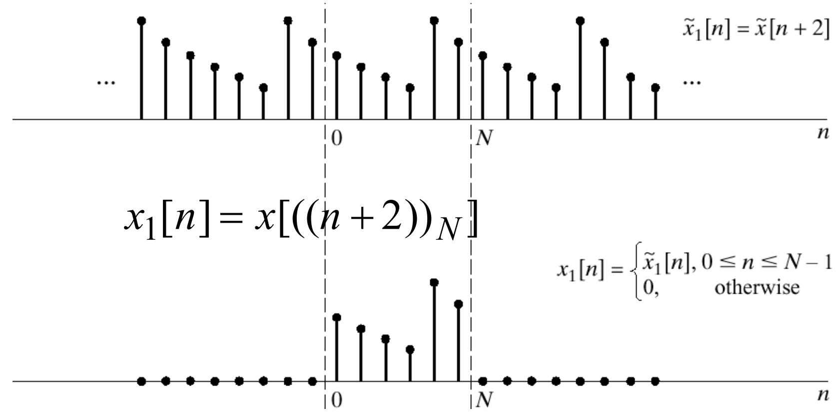
Circular Shift - I



$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN] = x[((n))_N]$$

Circular Shift - II

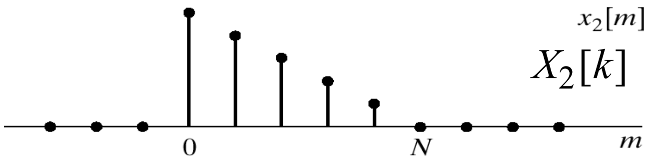
$$\tilde{x}_1[n] = \tilde{x}[n+2] = \sum_{r=-\infty}^{\infty} x[n+2+rN]$$

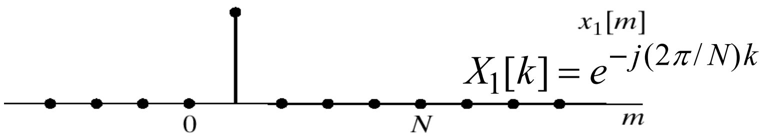


$$x_1[n] = x[((n+2))_N]$$

$$x_1[n] = \begin{cases} \tilde{x}_1[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Circular Convolution Example





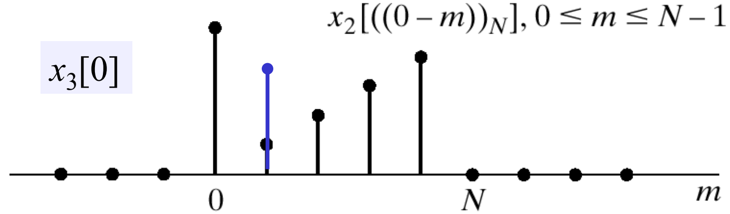
$$X_3[k] = X_1[k]X_2[k] = e^{-j(2\pi/N)k}X_2[k]$$

$$x_3[n] = \sum_{m=0}^{N-1} \delta[m-1]x_2[((n-m))_N] = x_2[((n-1))_N]$$

Circular Flipping and Shifting

$x_2[((0-m))_N], 0 \le m \le N-1$

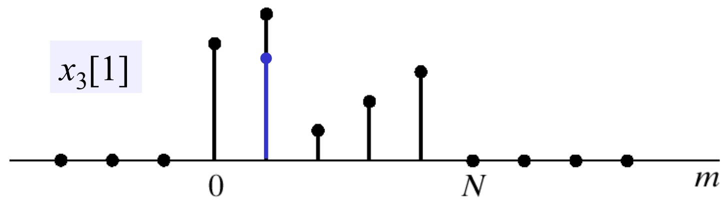
$x_3[0]$



$$x_3[n] = \sum_{m=0}^{N-1} \delta[m-1]x_2[((n-m))_N]$$

$x_2[((1-m))_N], 0 \le m \le N-1$

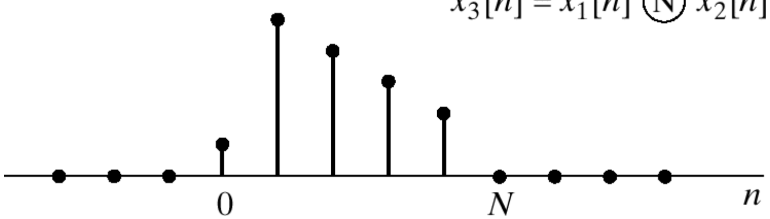
$x_3[1]$



Circular Convolution Example, concluded

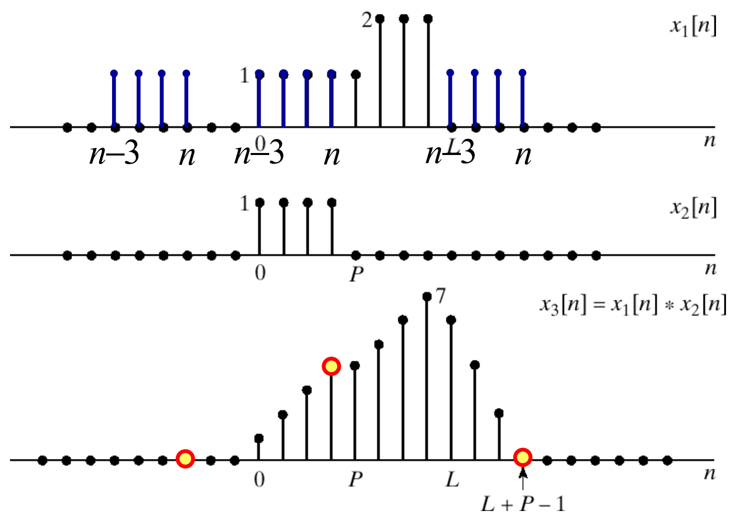
$$x_3[n] = \sum_{m=0}^{N-1} \delta[m-1]x_2[((n-m))_N]$$

$x_3[n] = x_1[n] \, \textcircled{N} \, x_2[n]$



$$x_3[n] = x_2[((n-1))_N]$$

Linear Convolution



If We Use a Big Enough DFT, We Can Compute a Linear Convolution …

Analysis for Convolution - 1

- If the DTFT of a sequence is the product of two other DTFTs (convolution):

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m] \Leftrightarrow X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$$

- … then its DFT will be the product of the corresponding DFTs:

$$X_3(e^{j(2\pi/N)k}) = X_1(e^{j(2\pi/N)k})X_2(e^{j(2\pi/N)k})$$

$$\Rightarrow X_3[k] = X_1[k]X_2[k]$$

Analysis for Convolution - 2

- The sequence obtained by an inverse N-point DFT will be periodic (repeated) with period N:

$$\tilde{x}_3[n] = \sum_{r=-\infty}^{\infty} x_3[n-rN] = x_3[n \text{ modulo } N] = x_3[((n))_N]$$

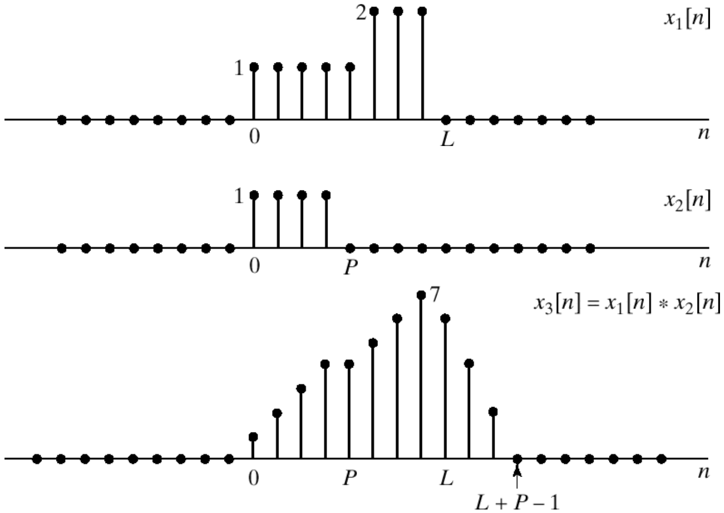
- We can pick out the desired linear convolution from the principal period:

$$x_3[n] = \tilde{x}_3[n], \quad n = 0, \ldots N-1$$

- … provided that the replicas do not overlap, *i.e.* $N \geq L+P-1$

In an Aliased Convolution, Some Parts of the Output Equal the Linear Convolution, and Some Don't …

Linear Convolution Example Again



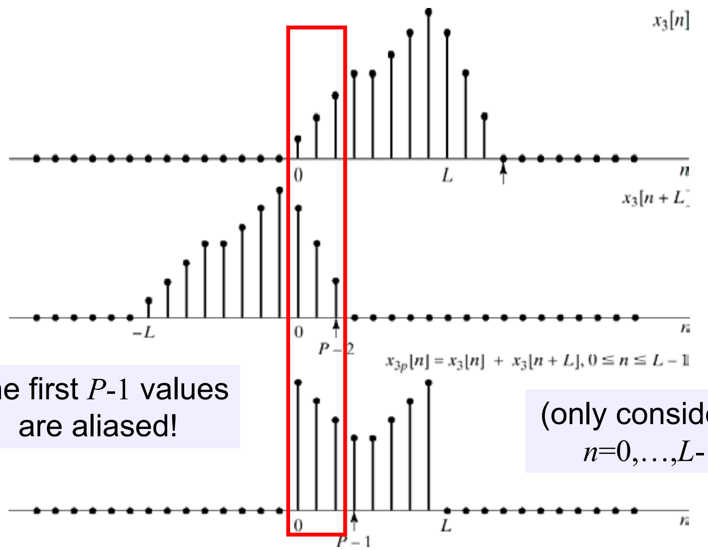
Aliased Convolution - 1

- If we tried to implement the convolution on the previous slide by multiplying the L-point DFTs of the two sequences, the output would be the linear convolution, but repeated (aliased) every L samples:

$$\tilde{x}_3[n] = \sum_{r=-\infty}^{\infty} x_3[n - rN] = x_3[n \text{ modulo } N] = x_3[((n))_N]$$

- If we examine the values in the principal interval of $n=0,\ldots,L\text{-}1$, what do we find?

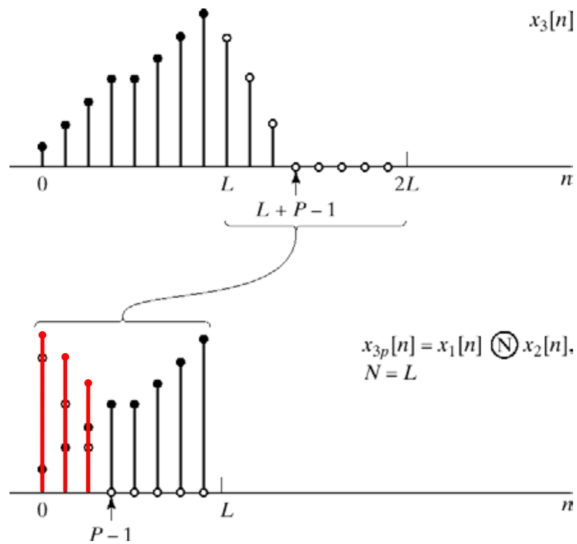
Aliased Convolution - 2



The first P-1 values are aliased!

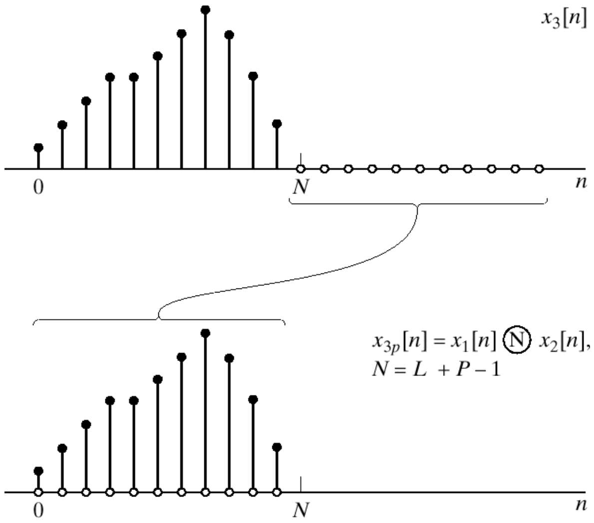
(only considering $n=0,\ldots,L\text{-}1$)

Aliased Convolution - 3



Another way to view it: the second block of L samples is aliased back onto the first block of L samples

Aliased Convolution - 4



If the DFT is big enough, namely $N \geq L+P-1$, then the "aliasing" simply overlaps zeroes onto the principal interval