ECE 4270

Fundamentals of Digital Signal Processing

Lecture 23: Discrete Spectrum Analysis

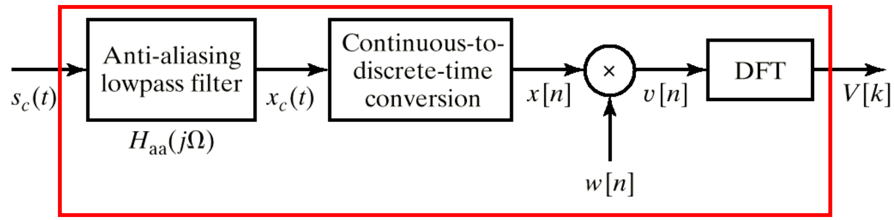
School of Electrical and Computer Engineering
Georgia Institute of Technology
Summer 2004

Spectrum Analysis of Analog Signals Using the Discrete Fourier Transform

Spectral Analysis

- To find energy or power distribution pattern over a range of frequencies
- To provide information for understanding or diagnosis of the signal (and the system that produces it)
- Knowledge of the signal provides hint on processing needs; e.g., presence of high frequency noise or hum from AC power or RF interference, harmonic patterns of instruments, etc.)
- A signal, deterministic or random, has a spectrum which is a function of frequency, implying that the signal consists of sinusoids
- The analysis result, depending on the assumptions (deterministic or random, stationary or non-stationary, etc.), may have different mathematical implications

Discrete Fourier Analysis of Analog Signals



- It is important to understand the following issues:
  - Anti-aliasing filtering: $S_c(j\Omega) \to X_c(j\Omega)$
  - C/D (actually A/D) conversion: $X_c(j\Omega) \to X(e^{j\omega})$
  - Windowing: $X(e^{j\omega}) \to V(e^{j\omega})$
  - Relationship of DFT to the DTFT: $V(e^{j\omega}) \to V[k]$

DT Fourier Analysis of CT Signals - 1

- Anti-aliasing filtering gives

$$x_c(t) = s_c(t) * h_{aa}(t) \Leftrightarrow X_c(j\Omega) = S_c(j\Omega) H_{aa}(j\Omega)$$

- If sampling occurs with no aliasing, then

$$x[n] = x_c(nT) \quad \Leftrightarrow \quad X(e^{j\omega}) = \frac{1}{T} X_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

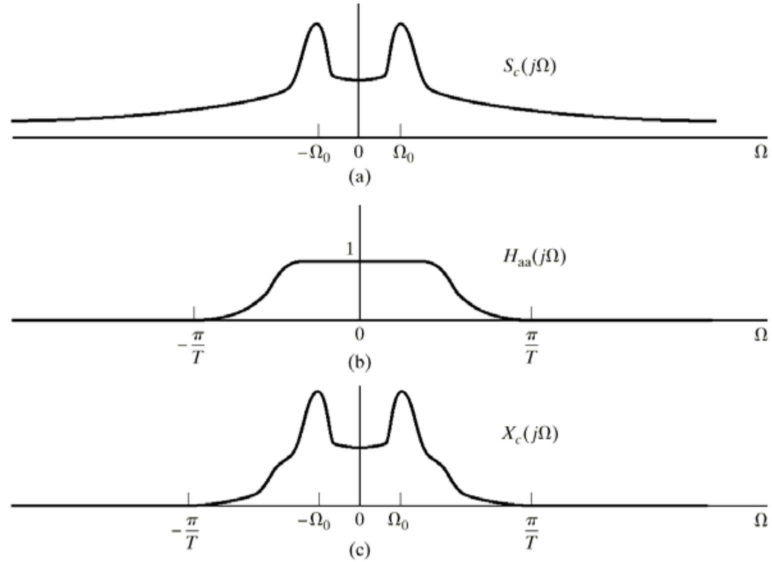
- Selecting a finite segment using a window gives

$$v[n] = w[n]x[n] \Leftrightarrow V(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta$$

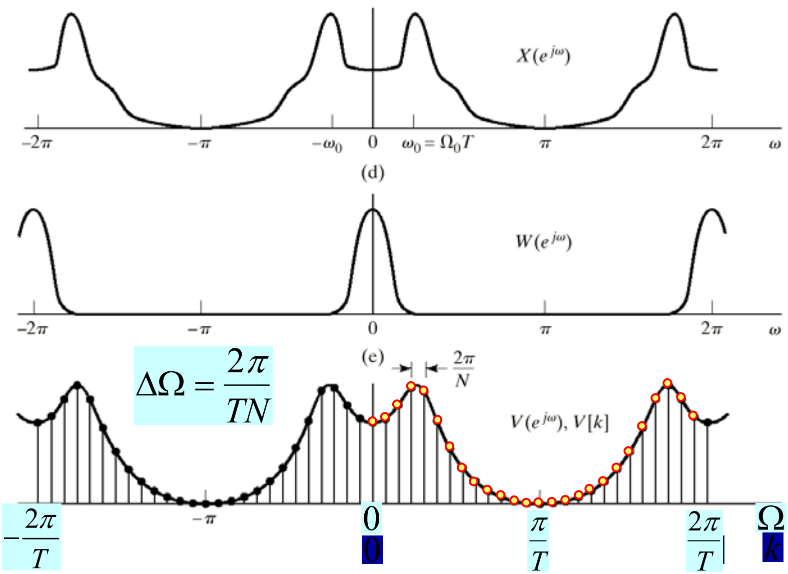
- The DFT of the windowed sequence is

$$V[k] = V(e^{j\omega_k})\Big|_{\omega_k = \frac{2\pi k}{N}} = V(e^{j\Omega_k T})\Big|_{\Omega_k = \frac{2\pi k}{NT}}, \quad k = 0, 1, \ldots, N-1.$$

DT Fourier Analysis of CT Signals - 2



DT Fourier Analysis of CT Signals - 3



Sinusoidal Signals - 1

- Continuous-time sinusoidal signal

$$s_c(t) = A_0\cos(\Omega_0 t + \theta_0) + A_1\cos(\Omega_1 t + \theta_1), \qquad -\infty < t < \infty$$

- Discrete-time sinusoid obtained by sampling

$$x[n] = A_0\cos(\omega_0 n + \theta_0) + A_1\cos(\omega_1 n + \theta_1), \qquad -\infty < n < \infty$$

$$\omega_0 = \Omega_0 T \text{ and } \omega_1 = \Omega_1 T$$

- Windowed discrete-time sinusoid (using Euler)

$$v[n] = \frac{A_0}{2} w[n] e^{j\theta_0} e^{j\omega_0 n} + \frac{A_0}{2} w[n] e^{-j\theta_0} e^{-j\omega_0 n} + \frac{A_1}{2} w[n] e^{j\theta_1} e^{j\omega_1 n} + \frac{A_1}{2} w[n] e^{-j\theta_1} e^{-j\omega_1 n}$$

Sinusoidal Signals - 2

- Windowed discrete-time sinusoid

$$v[n] = \frac{A_0}{2}w[n]e^{j\theta_0}e^{j\omega_0 n} + \frac{A_0}{2}w[n]e^{-j\theta_0}e^{-j\omega_0 n} + \frac{A_1}{2}w[n]e^{j\theta_1}e^{j\omega_1 n} + \frac{A_1}{2}w[n]e^{-j\theta_1}e^{-j\omega_1 n}$$

- DTFT of windowed discrete-time sinusoid

$$V(e^{j\omega}) = \frac{A_0}{2}e^{j\theta_0}W(e^{j(\omega-\omega_0)}) + \frac{A_0}{2}e^{-j\theta_0}W(e^{j(\omega+\omega_0)}) + \frac{A_1}{2}e^{j\theta_1}W(e^{j(\omega-\omega_1)}) + \frac{A_1}{2}e^{-j\theta_1}W(e^{j(\omega+\omega_1)})$$

- DFT of windowed discrete-time sinusoid

$$V[k] = V(e^{j\omega})\big|_{\omega=2\pi k/N}$$

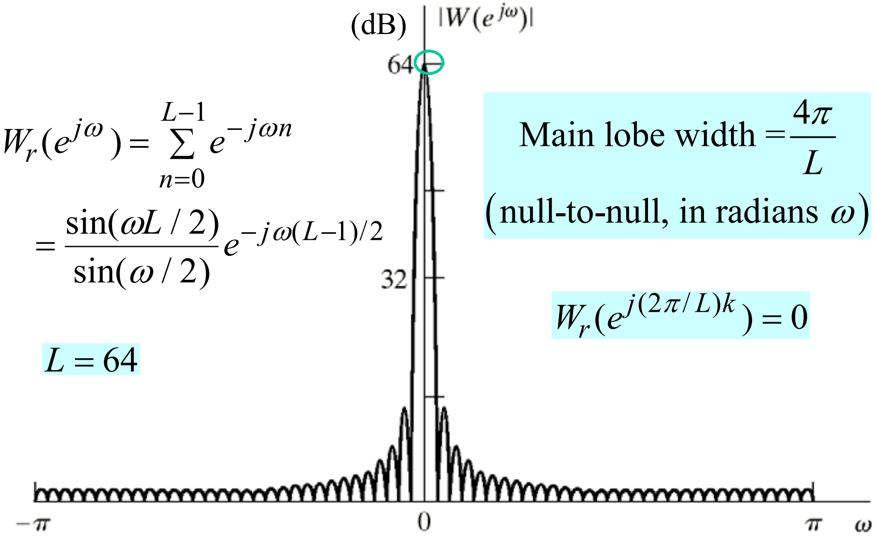
DTFT of Rectangular Window

$$W_r(e^{j\omega}) = \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

$L = 64$

Main lobe width $= \frac{4\pi}{L}$ (null-to-null, in radians ω)

$W_r(e^{j(2\pi/L)k}) = 0$



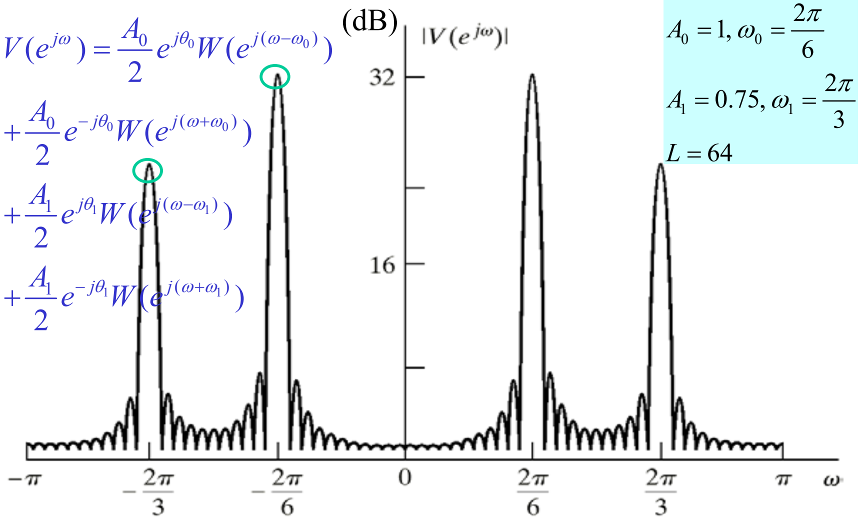
DTFT of Windowed Sinusoidal Signal - 1

$$V(e^{j\omega}) = \frac{A_0}{2}e^{j\theta_0}W(e^{j(\omega-\omega_0)}) + \frac{A_0}{2}e^{-j\theta_0}W(e^{j(\omega+\omega_0)}) + \frac{A_1}{2}e^{j\theta_1}W(e^{j(\omega-\omega_1)}) + \frac{A_1}{2}e^{-j\theta_1}W(e^{j(\omega+\omega_1)})$$

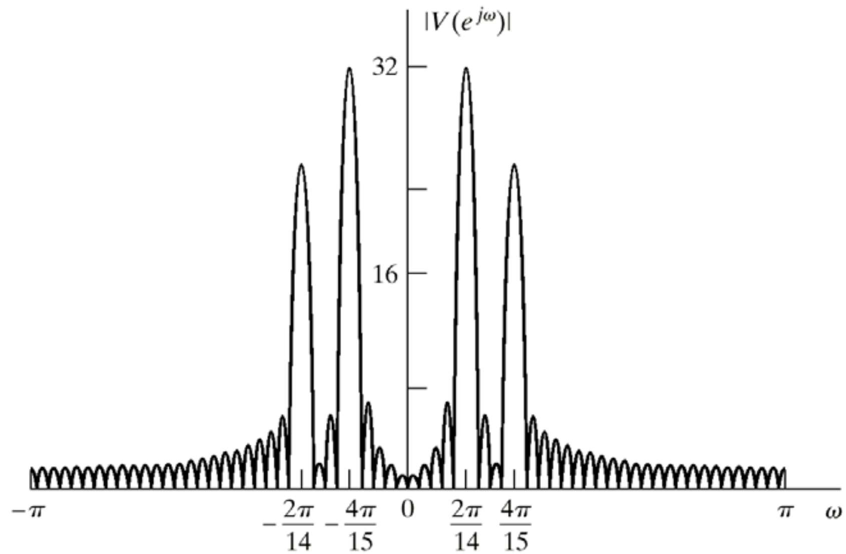
$A_0 = 1, \omega_0 = \frac{2\pi}{6}$

$A_1 = 0.75, \omega_1 = \frac{2\pi}{3}$

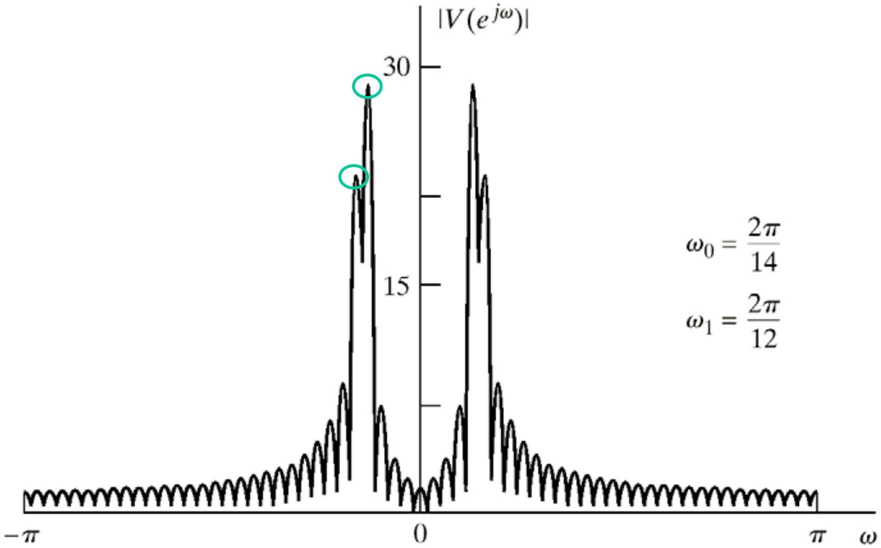
$L = 64$



DTFT of Windowed Sinusoidal Signal - 2



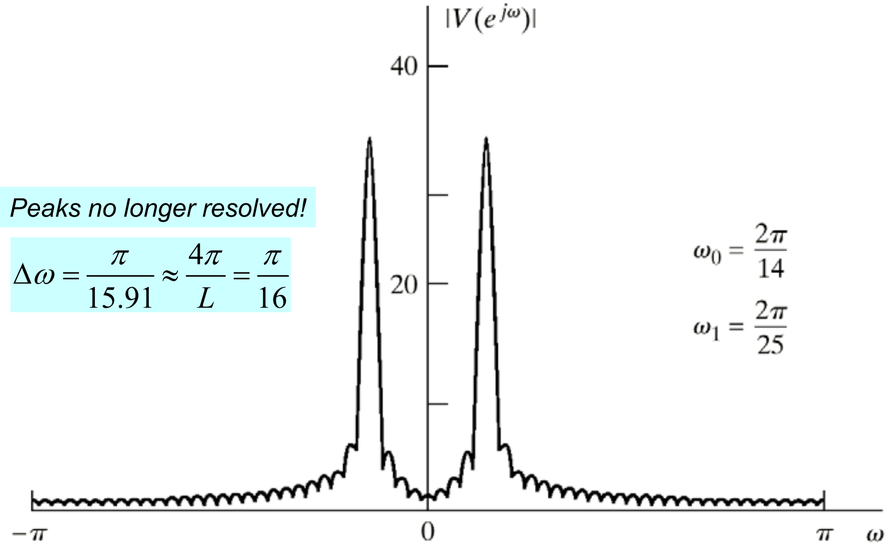
DTFT of Windowed Sinusoidal Signal - 3



DTFT of Windowed Sinusoidal Signal - 4

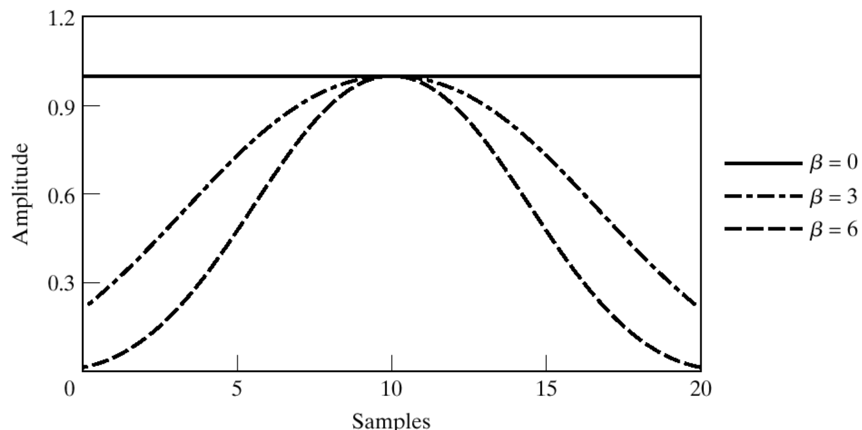
Peaks no longer resolved!

$$\Delta\omega = \frac{\pi}{15.91} \approx \frac{4\pi}{L} = \frac{\pi}{16}$$



We can reduce sidelobes at the expense of a widened mainlobe (poorer resolution) by windowing the data with other than a rectangular window …

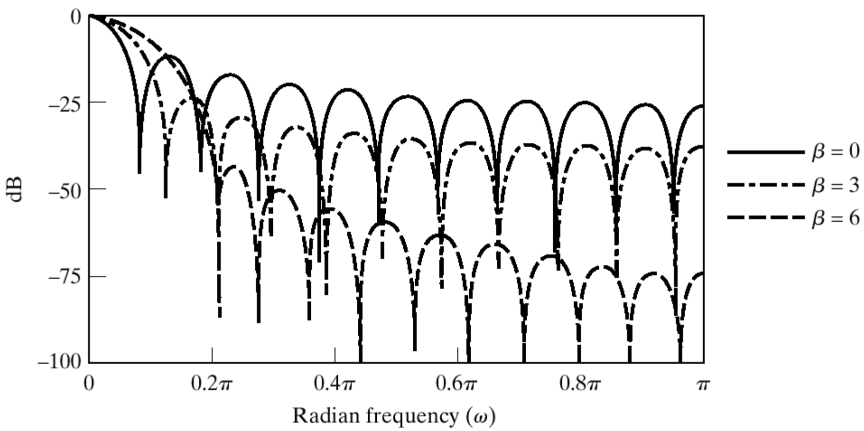
Kaiser Window



$$[\alpha = (L-1)/2]$$

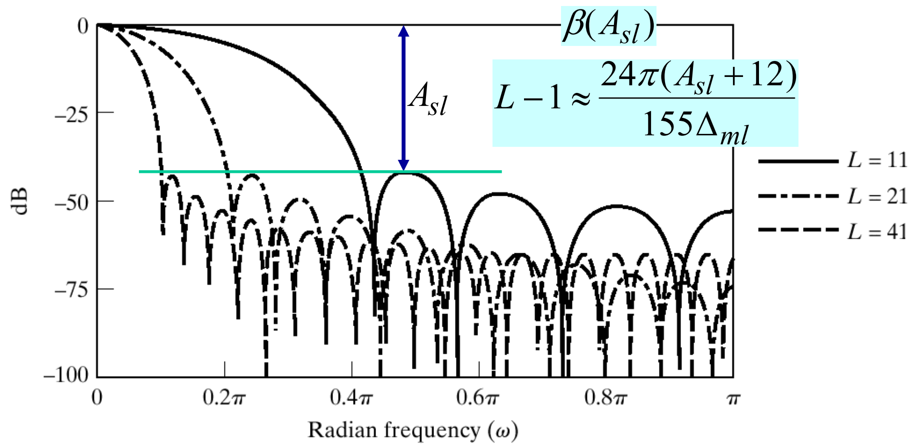
$$w_K[n] = \frac{I_0\left(\beta\sqrt{1-(n-\alpha)^2/\alpha^2}\right)}{I_0(\beta)} \quad 0 \le n \le L-1$$

DTFT of Kaiser Window - 1



Sidelobe height depends on β

DTFT of Kaiser Window - 2



Main lobe width Δ_{ml} is inversely proportional to L-1

Warning!

- Earlier in the semester, we had formulas for the Kaiser window that estimated the β and *order* M required to obtain a given *filter* sidelobe level and transition width when used in the window design method:

$$M = \frac{A-8}{2.285\Delta\omega} \qquad \beta = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 < A < 50 \\ 0.0 & A < 21 \end{cases}$$

- We now have *different* formulas that estimate β and *length* $L = M$-1 of the mainlobe width and sidelobes of *the DTFT of the window itself*:

$$L \cong \frac{24\pi\left(A_{sl}+12\right)}{155\Delta_{ml}} + 1$$

$$\beta = \begin{cases} 0, & A_{sl} < 13.26 \\ 0.76609(A_{sl}-13.26)^{0.4} + 0.09834(A_{sl}-13.26), & 13.26 < A_{sl} < 60 \\ 0.12438(A_{sl}+6.3), & 60 < A_{sl} < 120 \end{cases}$$

The Window Table Revisited

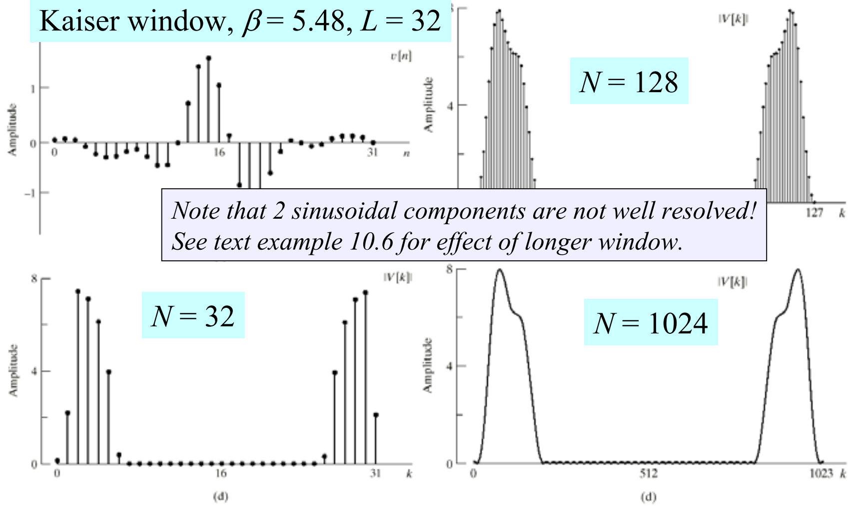
- We can also use non-parametric windows such as Hamming or Blackman for spectral analysis
  - now we care about the sidelobes of the DTFT of *the window itself*, not of filters designed with the window
  - one of the "bad" table columns is rehabilitated!

Type	PSL	Approx. Width of Mainlobe	Peak Approx. Error	Equivalent Kaiser Window β	Transition Width of Equivalent Kaiser Window
Rectangular	-13	$\frac{4\pi}{M+1}$	-21	0	$\frac{1.81\pi}{M}$
Bartlett	-25	$\frac{8\pi}{M}$	-25	1.33	$\frac{2.37\pi}{M}$
Hanning	-31	$\frac{8\pi}{M}$	-44	3.86	$\frac{5.01\pi}{M}$
Hamming	-41	$\frac{8\pi}{M}$	-53	4.86	$\frac{6.27\pi}{M}$
Blackman	-57	$\frac{12\pi}{M}$	-74	7.04	$\frac{9.19\pi}{M}$

DFT of Windowed Cosine Wave

$$x[n] = [\cos(2\pi n / 14) + 0.75\cos(4\pi n / 15)]w_K[n]$$

Kaiser window, $\beta = 5.48$, $L = 32$



Note that 2 sinusoidal components are not well resolved!
See text example 10.6 for effect of longer window.