ECE 4270

Fundamentals of Digital Signal Processing

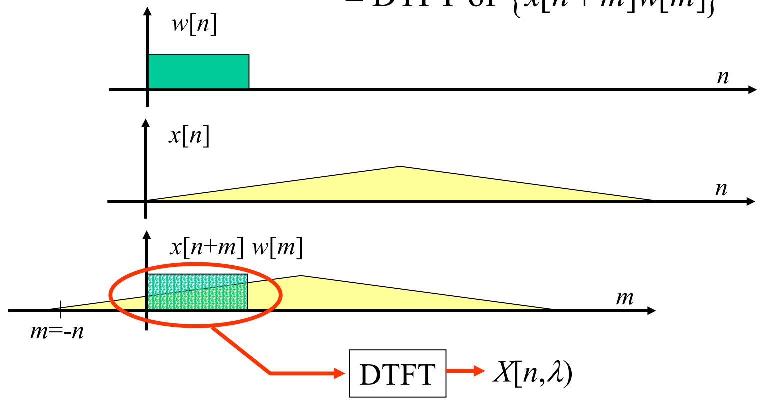
Lecture 24: Time Dependent Spectrum Analysis using Short-Time Fourier Transform

School of Electrical and Computer Engineering
Georgia Institute of Technology
Summer 2004

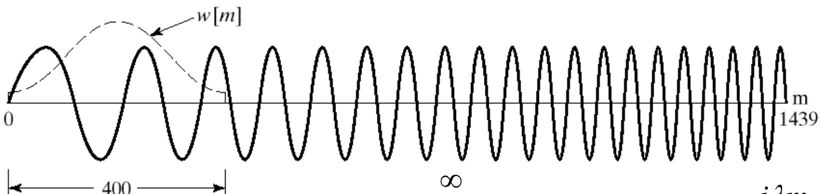
Time-Dependent Fourier Transform

- Definition: $X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m} \quad -\infty < n < \infty$

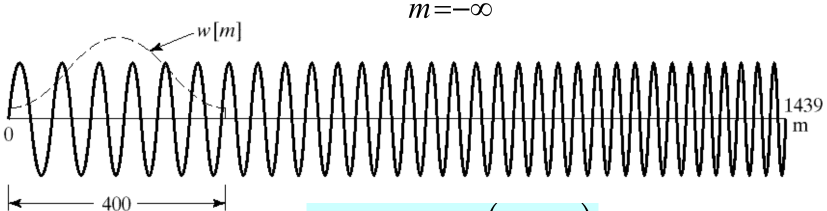
$$= \text{DTFT of } \{x[n+m]w[m]\}$$



Chirp Waveforms

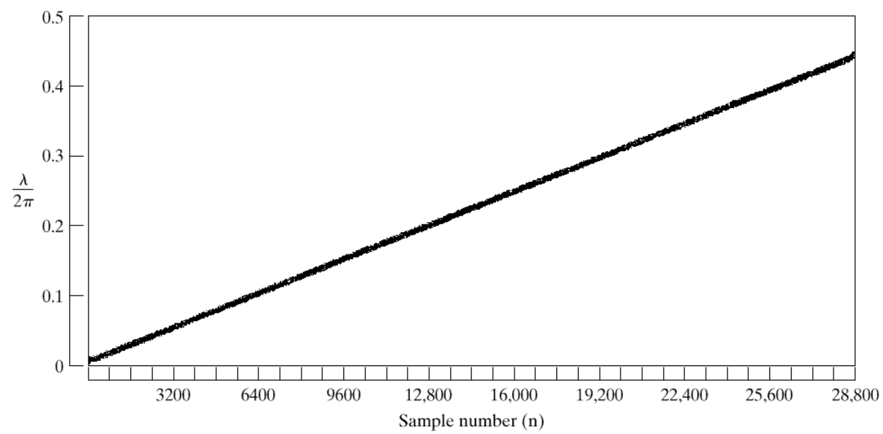


$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$



$$x[n] = \cos\left(\omega_0 n^2\right)$$

Spectrogram of Chirp Signal



TDFT as Linear Bandpass Filtering

- Definition:

$$X[n,\lambda)=\sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m} \qquad -\infty<n<\infty$$

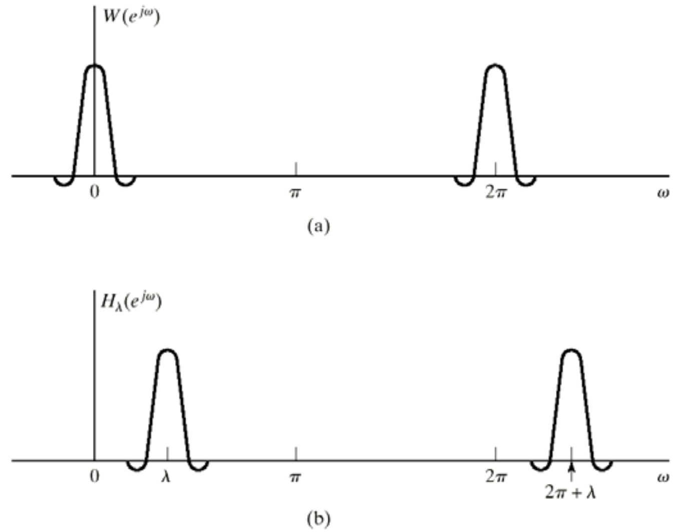
- Interpretation as linear bandpass filtering

$$\begin{aligned} X[n,\lambda) &= \sum_{\ell=-\infty}^{\infty} x[\ell]w[-(n-\ell)]e^{j\lambda(n-\ell)} \\ &= x[n]*h_\lambda[n] \end{aligned}$$

where

$$h_\lambda[n]=w[-n]e^{j\lambda n}\Leftrightarrow H_\lambda(e^{j\omega})=W(e^{j(\lambda-\omega)})$$

Linear Filtering Interpretations



Fourier transform of window (lowpass filter)

Bandpass filter for DTFT Frequency λ

Alternative TDFT Filtering Interpretation

- Interpretation as modulation, followed by lowpass filtering, and then followed again by modulation

$$\begin{aligned} X[n,\lambda) &= \sum_{\ell=-\infty}^{\infty} x[\ell]w[-(n-\ell)]e^{j\lambda(n-\ell)} \\ &= e^{j\lambda n}\left(\sum_{\ell=-\infty}^{\infty}\left(x[\ell]e^{-j\lambda\ell}\right)w[-(n-\ell)]\right) \end{aligned}$$

- We can show that the TDFT contains all the information required to reconstruct $x[n]$ for all n.

Effect of the Window in TDFT - 1

- Window parameters are length and shape
- These determine sidelobes and frequency resolving capability
  - Example for Kaiser:
    - Sidelobes determined by shape (β)

$$\beta=\begin{cases} 0, & A_{sl}<13.26 \\ 0.76609(A_{sl}-13.26)^{0.4}+0.09834(A_{sl}-13.26), & 13.26<A_{sl}<60 \\ 0.12438(A_{sl}+6.3), & 60<A_{sl}<120 \end{cases}$$

    - Resolution determined by length L:

$$L-1\cong\frac{24\pi\left(A_{sl}-12\right)}{155\Delta_{ml}}$$

Effect of the Window in TDFT - 2

- But window length also affects resolution in time!
  - longer windows fail to isolate short regions of "different" behavior
- So we have a tradeoff of resolution in time *vs.* resolution in frequency
  - "long" window: poor time resolution, good frequency resolution
  - "short" window: good time resolution, poor frequency resolution

Spectrum Analysis of Analog FM Signals

- A sinusoidally FM modulated cosine is

$$x_c(t) = \cos\theta(t) = \cos\big(\Omega_c t + (\Delta\Omega/\Omega_m)\sin\Omega_m t\big)$$

- Its *instantaneous frequency* is

$$\Omega_i(t) = \frac{d\theta(t)}{dt} = \Omega_c + \Delta\Omega\cos\Omega_m t$$

- Its Fourier transform is of the form

$$X_c(j\Omega) = \sum_{k=-\infty}^{\infty}[a_k\delta(\Omega-\Omega_c+k\Omega_m) + a_k^*\delta(\Omega+\Omega_c+k\Omega_m)]$$

- Carson's rule for bandwidth

$$\mathrm{BW} = 2(\Delta\Omega+\Omega_m)$$

Analysis of Discrete FM Signals

- Since the analog FM signal is bandlimited

$$x[n] = \cos\theta(nT) = \cos\big(\Omega_c nT + (\Delta\Omega/\Omega_m)\sin\Omega_m nT\big)$$

- Its *instantaneous frequency* is

$$\omega_i = \Omega_i(nT)T = \Omega_c T + \Delta\Omega T\cos\Omega_m nT$$

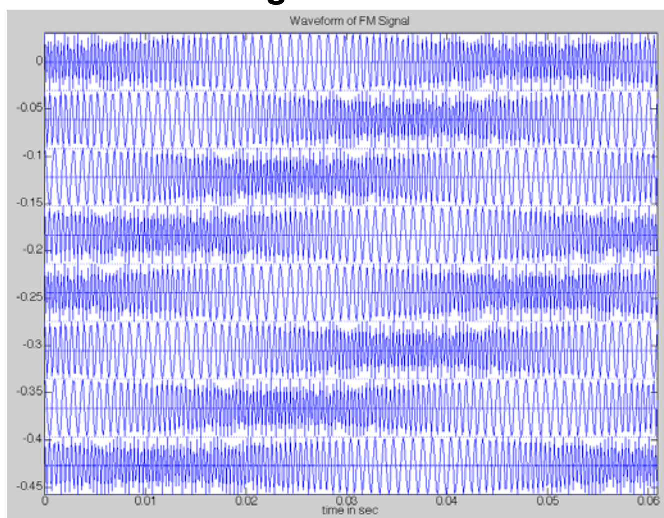
- Its Fourier transform is of the form

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T}+\frac{2\pi k}{T}\right)\right)$$

- Carson's rule for bandwidth (analog)

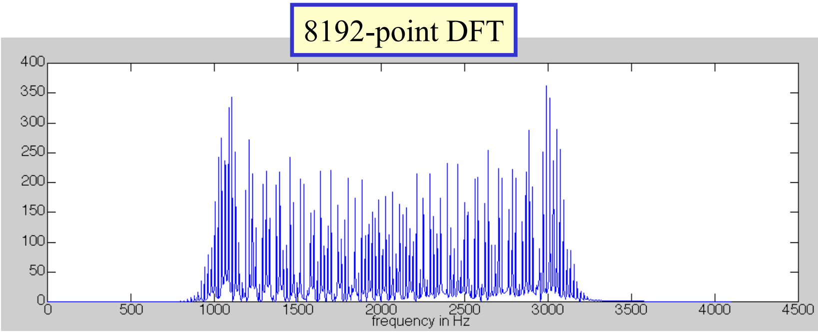
$$\mathrm{BW} = 2(\Delta\Omega+\Omega_m)T$$

FM Signal Waveform



$$\frac{1}{T} = 8192,\ \Omega_c = 2\pi(2048),\ \Omega_m = 2\pi(10.24),\ \Delta\Omega = 2\pi(1024)$$

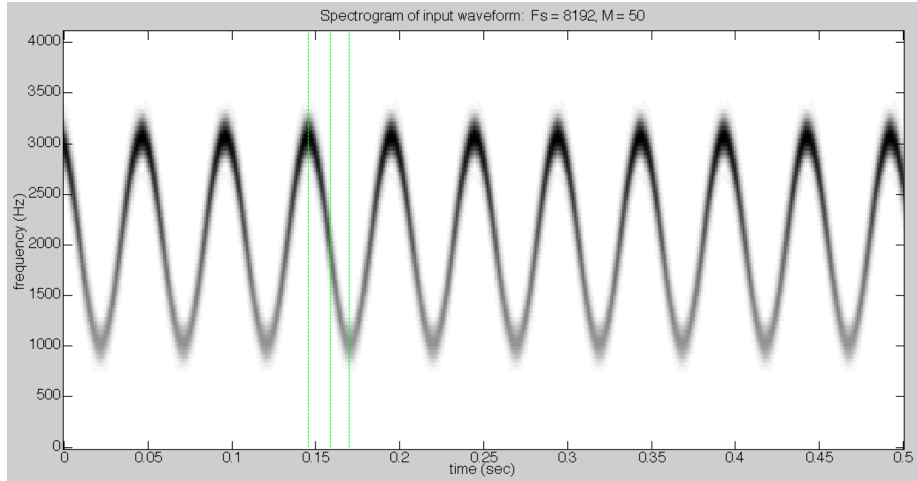
Long-Term Spectrum of FM Signal



Carson's rule for bandwidth gives a spectrum that is non-zero over the following band:

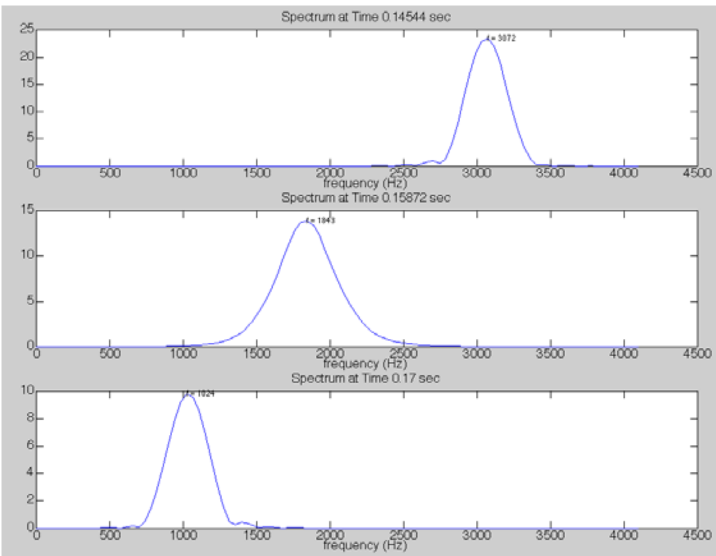
$$\Omega_c \pm BW/2 = \Omega_c \pm (\Delta\Omega + \Omega_m) = 2\pi(2048 \pm 1034.24)$$

Spectrogram of FM Signal

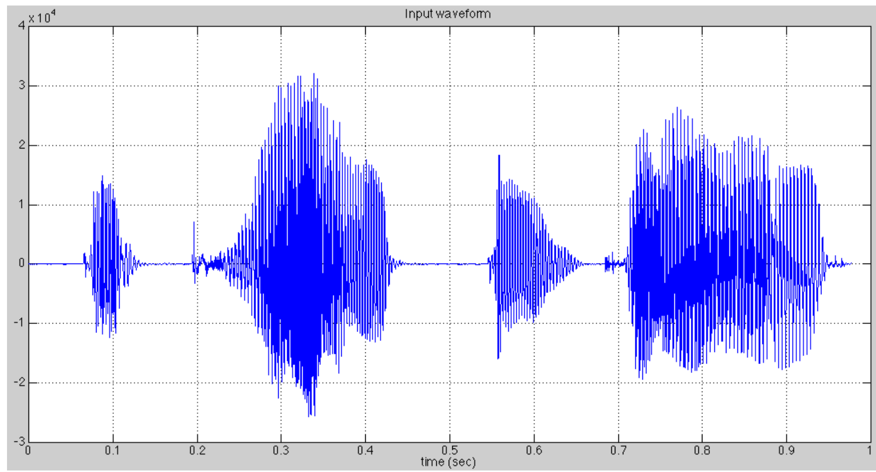


Window length is 50/8192 ≈ 6.1 msec

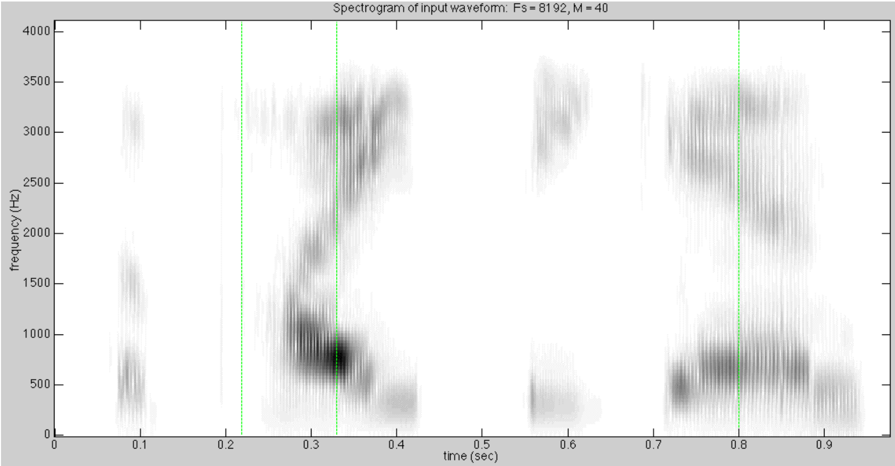
Slices of FM Spectrogram



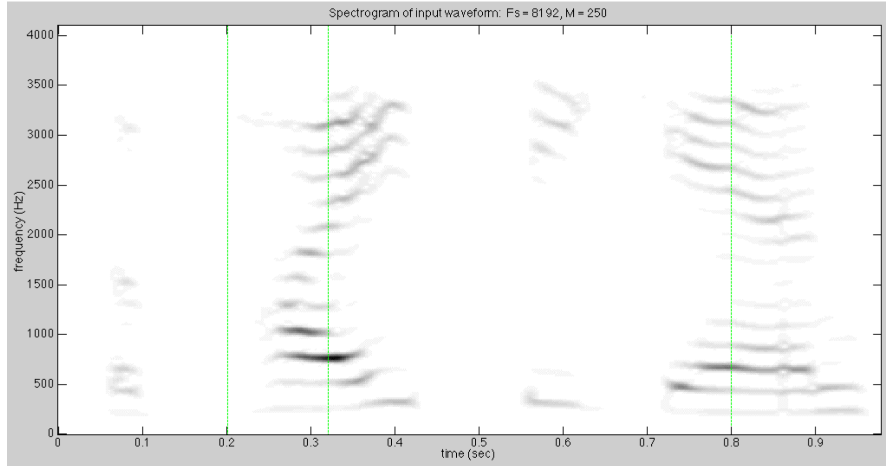
Speech Waveform



Wideband Spectrogram



Narrowband Spectrogram



Narrowband Slices