ECE 4270
Fundamentals of Digital Signal Processing

Lecture 3: Frequency Response, Discrete-Time Fourier Transform

School of Electrical and Computer Engineering
Georgia Institute of Technology
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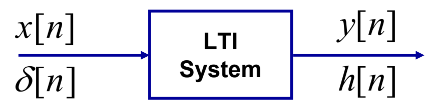
Review of LTI Systems

- Linear time-invariant systems are completely characterized by their responses to the unit impulse sequence.



convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n]*h[n] = h[n]*x[n]$$

$$\text{BIBO stability requires } \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$$\text{Causality requires } h[n] = 0 \quad \text{for} \quad n < 0$$

Difference Equations

- For all computationally realizable LTI systems, the input and output satisfy a difference equation of the form

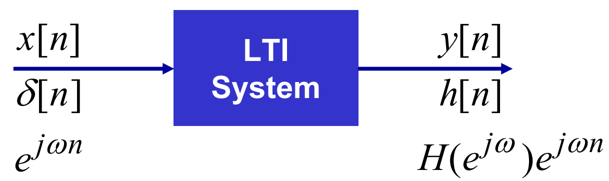
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- This leads to the recurrence formula

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k] + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-k]$$

which can be used to compute the "present" output from the present and M past values of the input and N past values of the output

Complex Exponential Input Signals



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

$$= H(e^{j\omega})e^{j\omega n}$$

Frequency response $H(e^{j\omega})$

Eigenfunctions of LTI Systems

- A signal $x[n]$ is an eigenfunction of a system T if

$$T\{x[n]\} = \lambda x[n]$$

where λ is a complex constant that can be different for each $x[n]$

- Thus complex exponentials are eigenfunctions of LTI systems since

$$T\{e^{j\omega n}\} = H(e^{j\omega})e^{j\omega n}$$

and the frequency response $H(e^{j\omega})$ is the associated eigenvalue

The Frequency Response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Periodicity of the frequency response:

$$H(e^{j(\omega+2\pi)}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j(\omega+2\pi)k} = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\underbrace{e^{-j2\pi k}}_{1}$$

- Convergence of the frequency response:

$$\left|H(e^{j\omega})\right| = \left|\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right| \le \sum_{k=-\infty}^{\infty} |h[k]|\underbrace{\left|e^{-j\omega k}\right|}_{1}$$

$$\left|H(e^{j\omega})\right| \le \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Same as condition for stability!

Delay and First Difference

- Delay: $y[n] = x[n-n_d]$ $\quad h[n] = \delta[n-n_d]$

$$x[n] = e^{j\omega n} \mapsto y[n] = e^{j\omega(n-n_d)} = \underbrace{e^{-j\omega n_d}}_{H(e^{j\omega})} e^{j\omega n}$$

$$H(e^{j\omega}) = e^{-j\omega n_d}$$

- First difference:

$$y[n] = x[n] - x[n-1] \qquad h[n] = \delta[n] - \delta[n-1]$$

$$x[n] = e^{j\omega n} \mapsto y[n] = e^{j\omega n} - e^{j\omega(n-1)} = \underbrace{(1-e^{-j\omega})}_{H(e^{j\omega})} e^{j\omega n}$$

$$H(e^{j\omega}) = 1 - e^{-j\omega}$$

More on the Ideal Delay

- The impulse response and frequency response of a pure delay are

$$h[n] = \delta[n-n_d] \Leftrightarrow H(e^{j\omega}) = e^{-j\omega n_d}$$

- Note that the magnitude of the frequency response is 1 for all ω

$$|H(\omega)| \equiv 1$$

and the phase is linear in ω

$$\arg\{H(\omega)\} = -\omega n_d$$

- Conversely, a frequency response with these characteristics corresponds to a pure delay

Moving Average

$$y[n] = \frac{1}{M+1}\sum_{k=0}^{M} x[n-k]$$

$$x[n] = e^{j\omega n} \mapsto y[n] = \frac{1}{M+1}\sum_{k=0}^{M} e^{j\omega(n-k)}$$

$$= \underbrace{\frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k}}_{H(e^{j\omega})} e^{j\omega n}$$

$$H(e^{j\omega}) = \frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k}$$

Useful Summation Formula

$$\sum_{n=N_1}^{N_2} \alpha^n = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

- Geometric series; finite sum, infinite sum
- Used for frequency response of moving average, for instance, in text example 2.20
  - *Study this in detail!*

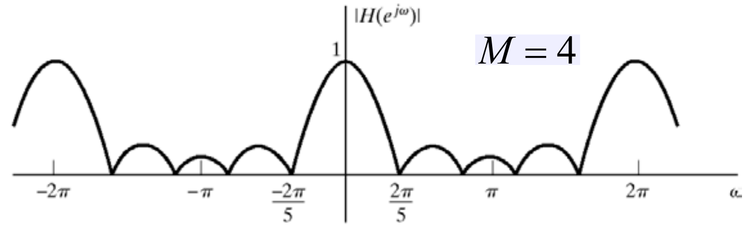
Moving Average Frequency Response

$$H(e^{j\omega}) = \frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k} = \frac{1}{M+1}\frac{(1-e^{-j\omega(M+1)})}{(1-e^{-j\omega})}$$

$$H(e^{j\omega}) = \frac{1}{M+1}\frac{\left\{e^{j\omega(M+1)/2} - e^{-j\omega(M+1)/2}\right\}e^{-j\omega(M+1)/2}}{(e^{j\omega/2} - e^{-j\omega/2})e^{-j\omega/2}}$$

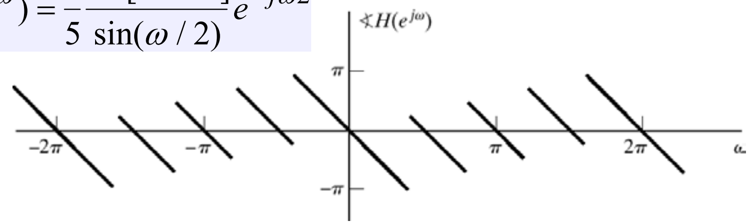
$$H(e^{j\omega}) = \frac{1}{M+1}\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$$

Plotting the Frequency Response



$M = 4$

$$H(e^{j\omega}) = \frac{1}{5}\frac{\sin[\omega 5/2]}{\sin(\omega/2)}e^{-j\omega 2}$$



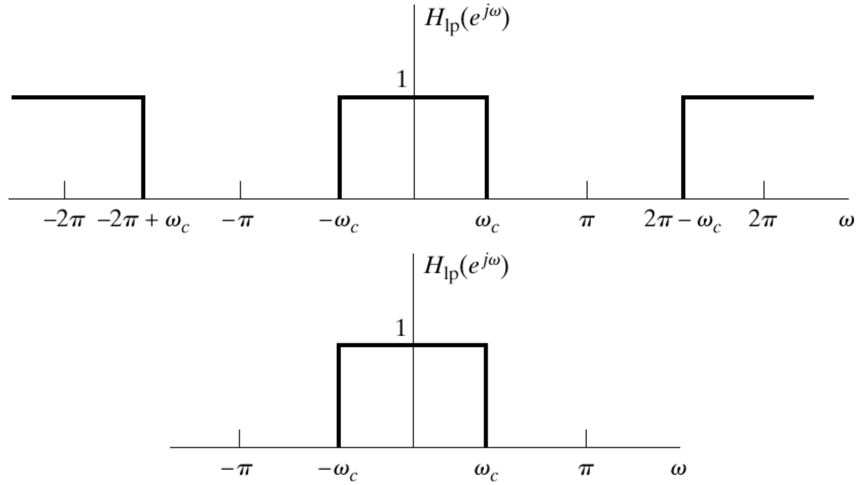
Plotting using MATLAB

»help freqz

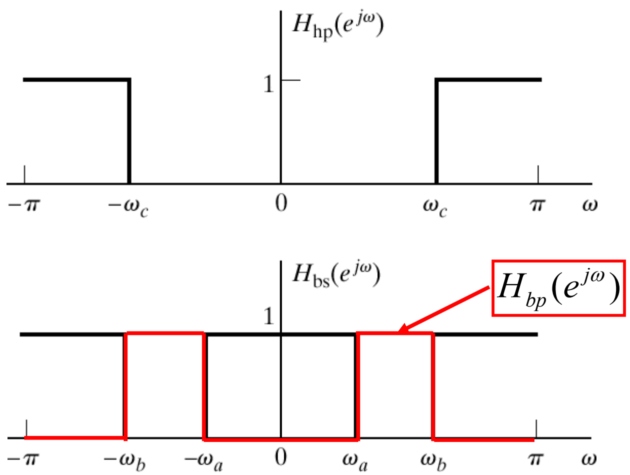
FREQZ Z-transform digital filter frequency response.
When N is an integer, [H,W] = FREQZ(B,A,N) returns the N-point frequency vector W in radians and the N-point complex frequency response vector H given numerator and denominator coefficients in vectors B and A. The frequency response is evaluated at N points equally spaced around the upper half of the unit circle. If N isn't specified, it defaults to 512.

```
>> omega=(0:400)*pi/400; b=[1,1,1,1,1]/5;
>> H=freqz(b,1,omega);
>> subplot(211); plot(omega/pi,abs(H))
>> subplot(212); plot(omega/pi,angle(H))
```


Ideal Lowpass Filter



Frequency Selective Filters



Discrete-Time Fourier Transform

- Definition:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ Direct Transform

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$ Inverse Transform

- Existence of the DTFT:

$$\left|X(e^{j\omega})\right| \le \sum_{n=-\infty}^{\infty} |x[n]|\overbrace{\left|e^{-j\omega n}\right|}^{1} \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Frequency Response Again

- The frequency response function is now seen to be just the DTFT of the impulse response.

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Note that stable systems have frequency responses $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$
- Therefore, the impulse response is just the inverse DTFT of the frequency response.

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

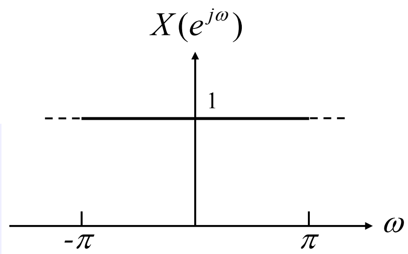
DTFT of Unit Impulse

Direct Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = 1 \qquad \text{for all } \omega$$

Inverse Transform

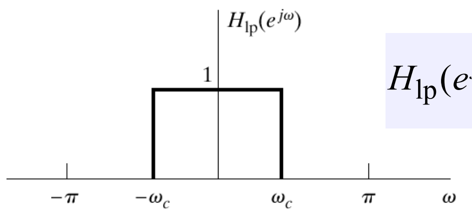
$$\delta[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \times e^{j\omega n} d\omega$$



A unit impulse consists of an ensemble of complex exponentials (sinusoids) of every frequency between $-\pi$ and π.

Try to understand why analysis of LTI was done with complex exponentials

Ideal Lowpass Filter



$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_{\mathrm{lp}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{lp}}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

$$= \frac{1}{j2\pi n}\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{j2\pi n}$$

$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c n}{\pi n} \qquad -\infty < n < \infty$$

Example

- Consider the real exponential signal

$$x[n] = a^n u[n]$$

- Its DTFT is obtained as follows:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1 - ae^{j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})}$$

$$= \underbrace{\frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega}}_{X_R(e^{j\omega})} + j\underbrace{\frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega}}_{X_I(e^{j\omega})}$$

Magnitude and Angle Form

$$x[n] = a^n u[n] \quad \Leftrightarrow \quad X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\angle X(e^{j\omega})}$$

$$\left|X(e^{j\omega})\right|^2 = X(e^{j\omega})X^*(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})(1 - ae^{j\omega})}$$

$$= \frac{1}{1 + a^2 - 2a\cos\omega}$$

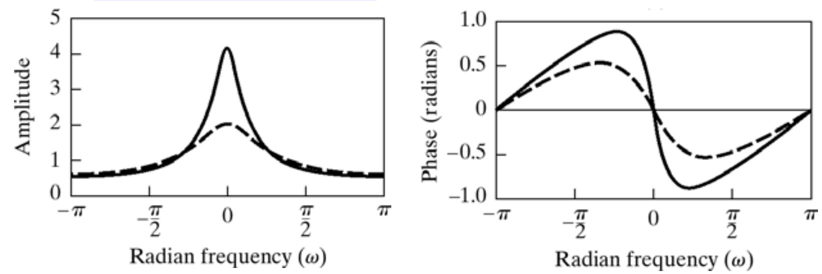
$$\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right)$$

Magnitude and Angle Plots

$$\left|X(e^{j\omega})\right| = \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}}$$

$$\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right)$$

$$\left|X(e^{j\omega})\right| = \left|X(e^{-j\omega})\right| \qquad \angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$$



DTFT Symmetry Properties - 1

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$

DTFT Symmetry Properties - 2

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Another Example

- Consider a signal $x[n] = Ar^n \cos(\omega_0 n + \phi)u[n]$
- To find its DTFT, break it up into complex exponentials

$$x[n] = \frac{A}{2}e^{j\phi}r^n e^{j\omega_0 n}u[n] + \frac{A}{2}e^{-j\phi}r^n e^{-j\omega_0 n}u[n]$$

- Therefore, the DTFT is

$$X(e^{j\omega}) = \frac{\frac{A}{2}e^{j\phi}}{1 - re^{j\omega_0}e^{-j\omega}} + \frac{\frac{A}{2}e^{-j\phi}}{1 - re^{-j\omega_0}e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{A\cos\phi - Ar\cos(\omega_0 - \phi)e^{-j\omega}}{1 - 2r\cos\omega_0 e^{-j\omega} + r^2 e^{-j2\omega}}$$

Using the DTFT

- The DTFT provides a “frequency-domain” representation that is invaluable for thinking about and solving DSP problems.
- To use it effectively you must
  – know the Fourier transforms of certain important signals
  – know its properties and certain key theorems
  – be able to combine time-domain and frequency domain methods appropriately

Fourier Transform Pairs - 1

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n - n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$

We worked this one out.

Fourier Transform Pairs - 2

Sequence	Fourier Transform
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p}u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

And this one too.

The Unit Impulse "Function"

- The continuous-variable ("Dirac") impulse is defined by the following properties:

$$\delta(\omega) = 0 \;\text{ for }\; \omega \neq 0 \qquad \text{highly concentrated}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(\omega)d\omega = 1 \;\text{ for }\; \varepsilon > 0 \qquad \text{area concentrated}$$

$$X(e^{j\omega})\delta(\omega) = X(e^{j0})\delta(\omega) \quad \text{sampling property}$$

$$\delta(\omega) * X(e^{j\omega}) = X(e^{j\omega}) \;\; \text{replicating property}$$

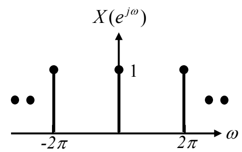
DTFT of a Constant Signal

- Assume a "periodic impulse train" for the DTFT

$$x[n] = 1 \Leftrightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

- Find the corresponding sequence $x[n]$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$



"DC" signal

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)\right\}e^{j\omega n}d\omega$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi} \cancel{2\pi}\delta(\omega)e^{j\omega n}d\omega = \int_{-\pi}^{\pi}\delta(\omega)e^{j0n}d\omega = 1$$

Very Useful DTFT Pairs

$$x[n] = \delta[n - n_d] \Leftrightarrow X(e^{j\omega}) = e^{-j\omega n_d}$$

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$x[n] = 1 \Leftrightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

$$x[n] = \frac{\sin \omega_c n}{\pi n} \Leftrightarrow X(e^{j\omega}) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| > \omega_c \end{cases}$$

$$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow X(e^{j\omega}) = \frac{\sin[(M+1)\omega/2]}{\sin(\omega/2)}e^{-j\omega M}$$

Fourier Transform Theorems - 1

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$

Fourier Transform Theorems - 2

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

DTFT of Sinusoids

- Recall that

$$x[n]=1 \Leftrightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega+2\pi r)$$

- Also note that

$$x[n]e^{j\omega_0 n} \Leftrightarrow X(e^{j(\omega-\omega_0)})$$

- Therefore

$$e^{j\omega_0 n} \Leftrightarrow \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi r)$$

$$\cos\omega_0 n \Leftrightarrow \sum_{r=-\infty}^{\infty} \pi\delta(\omega+\omega_0+2\pi r)+\pi\delta(\omega-\omega_0+2\pi r)$$

Convolution Theorem

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k] \Leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$x[n]$ → **LTI System** → $y[n]$

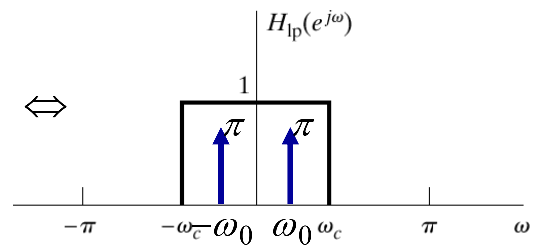
$\delta[n]$ → $h[n]$

$e^{j\omega n}$ → $H(e^{j\omega})e^{j\omega n}$

$X(e^{j\omega})$ → $X(e^{j\omega})H(e^{j\omega})$

Example 1

$$h_{lp}[n]=\frac{\sin\omega_c n}{\pi n} \Leftrightarrow$$



- Find the output when the input is

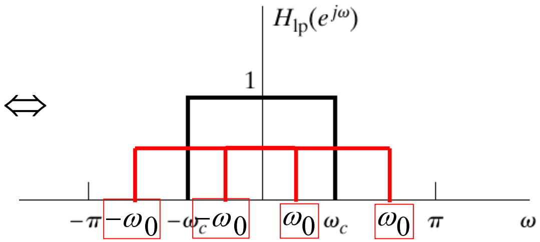
$x[n]=\cos\omega_0 n \Leftrightarrow$

$$X(e^{j\omega})=\sum_{r=-\infty}^{\infty} \pi\delta(\omega+\omega_0+2\pi r)+\pi\delta(\omega-\omega_0+2\pi r)$$

$$y[n]=\cos\omega_0 n \quad \text{if} \quad \omega_0<\omega_c$$

Example 2

$$h_{\rm lp}[n] = \frac{\sin \omega_c n}{\pi n} \quad \Leftrightarrow$$



- Find the output when the input is

$$x[n] = \frac{\sin(\omega_0 n)}{2\pi n}$$

$$\boxed{\mathbf{y[n]} = \frac{\sin(\omega_0 n)}{2\pi n} \text{ if } \omega_c \gtrless \omega_0}$$

Frequency Response of a DE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k Y(e^{j\omega}) e^{-j\omega k} = \sum_{k=0}^{M} b_k X(e^{j\omega}) e^{-j\omega k}$$

$$\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right) X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right)}{\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right)}$$

Example 3

- Suppose that the difference equation is

$$y[n] = y[n-1] - .9y[n-2] + x[n] + x[n-1]$$

- The frequency response is

$$H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - e^{-j\omega} + .9e^{-j\omega 2}}$$

- This system is implemented in MATLAB by

```
>> y=filter([1,1],[1,-1,.9],x)
```

We can compute its frequency response by

```
>> omega=(0:500)*pi/500;
>> H=freqz ([1,1],[1,-1,.9],omega);
```