ECE 4270

Fundamentals of Digital Signal Processing

Lecture 4: Fourier Transforms, Random Signals

School of Electrical and Computer Engineering
Georgia Institute of Technology
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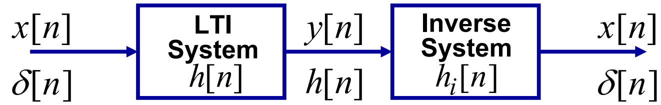
Inverse Systems

- An inverse system compensates (undoes) the effects of another system.



- The accumulator and first-difference systems are inverses of each other.

$$\Rightarrow h[n] * h_i[n] = \delta[n]$$

- Understanding inverse systems is greatly facilitated by transform methods.

$$(\delta[n] - \delta[n-1]) * u[n] = u[n] - u[n-1] = \delta[n]$$

Very Useful DTFT Pairs

$$x[n] = \delta[n - n_d] \Leftrightarrow X(e^{j\omega}) = e^{-j\omega n_d}$$

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$x[n] = 1 \Leftrightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

$$x[n] = \frac{\sin \omega_c n}{\pi n} \Leftrightarrow X(e^{j\omega}) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| > \omega_c \end{cases}$$

$$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow X(e^{j\omega}) = \frac{\sin[(M+1)\omega/2]}{\sin(\omega/2)} e^{-j\omega M}$$

Fourier Transform Theorems - 1

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$

Fourier Transform Theorems - 2

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

DTFT of Sinusoids

- Recall that

$$x[n]=1 \Leftrightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega+2\pi r)$$

- Also note that

$$x[n]e^{j\omega_0 n} \Leftrightarrow X(e^{j(\omega-\omega_0)})$$

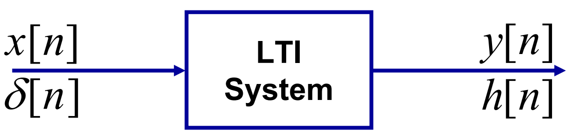
- Therefore

$$e^{j\omega_0 n} \Leftrightarrow \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi r)$$

$$\cos\omega_0 n \Leftrightarrow \sum_{r=-\infty}^{\infty} \pi\delta(\omega+\omega_0+2\pi r)+\pi\delta(\omega-\omega_0+2\pi r)$$

Convolution Theorem

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k] \Leftrightarrow Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})$$

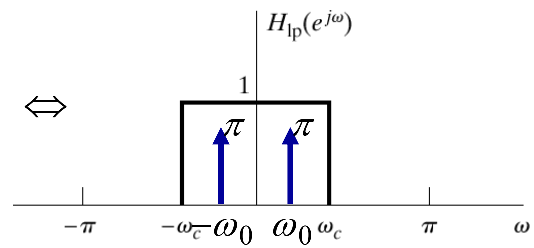


$e^{j\omega n}$ $\quad$ $H(e^{j\omega})e^{j\omega n}$

$X(e^{j\omega})$ $\quad$ $X(e^{j\omega})H(e^{j\omega})$

Example 1

$$h_{lp}[n]=\frac{\sin\omega_c n}{\pi n} \Leftrightarrow$$



- Find the output when the input is

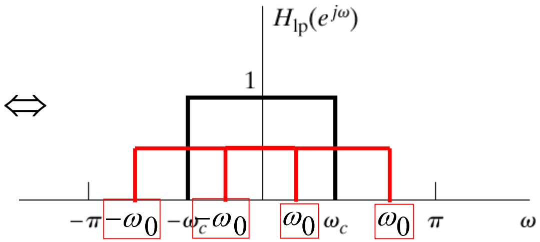
$x[n]=\cos\omega_0 n \Leftrightarrow$

$$X(e^{j\omega})=\sum_{r=-\infty}^{\infty} \pi\delta(\omega+\omega_0+2\pi r)+\pi\delta(\omega-\omega_0+2\pi r)$$

$$y[n]=\cos\omega_0 n \quad \text{if } \omega_0<\omega_c$$

Example 2

$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c n}{\pi n} \quad \Leftrightarrow$$



- Find the output when the input is

$$x[n] = \frac{\sin(\omega_0 n)}{2\pi n}$$

$$\boldsymbol{y[n] = \frac{\sin(\omega_0 n)}{2\pi n}} \text{ if } \omega_c \geq \omega_0$$

Frequency Response of a DE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k Y(e^{j\omega}) e^{-j\omega k} = \sum_{k=0}^{M} b_k X(e^{j\omega}) e^{-j\omega k}$$

$$\left(\sum_{k=0}^{N} a_k e^{-j\omega k} \right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} b_k e^{-j\omega k} \right) X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\omega k} \right)}{\left(\sum_{k=0}^{N} a_k e^{-j\omega k} \right)}$$

Frequency Response, continued

- The frequency response of a linear time-invariant system provides a frequency-dependent weighting on the signal.
- The power spectrum of a signal is a manifestation of the energy concentration, per unit time, of the signal at various frequencies.
- An LTI system modifies the power spectrum of the input signal by multiplying it with the frequency-dependent weighting.
- How to compute power spectrum?
  - For a deterministic signal
  - For a random signal

What is a random signal?

- Many signals vary in complicated and uncertain patterns that cannot be easily described by simple equations.
  - The uncertainty may be caused by interferences such as observation noise or fluctuation in system parameters.
  - It is often convenient and useful to consider such signals as being created or influenced by some sort of random mechanism.
- The mathematical representation of “random signals” involves the concept of a *random process*.

Random Variables

A (real) random variable is a random event defined on the real line, characterized by a probability assignment (a number between 0 and 1), or a probability distribution

$$P_{\mathbf{x}}(x) = \Pr\{\mathbf{x} \le x\} = \int_{-\infty}^{x} p_{\mathbf{x}}(\theta)d\theta$$

where

$p_{\mathbf{x}}(x) = \dfrac{\partial P_{\mathbf{x}}(x)}{\partial x}$ is the corresponding probability **density** function, assuming $P_{\mathbf{x}}$ is differentiable.

Random Process

- A discrete-time random process is an indexed sequence of random variables $\{\mathbf{x}_m\}, -\infty < m < \infty$ characterized by joint distribution

$$P_{\cdots,\mathbf{x}_m,\mathbf{x}_{m+1},\cdots}(\cdots,x_m,x_{m+1},\cdots) = \Pr\{\cdots,\mathbf{x}_m \le x_m, \mathbf{x}_{m+1} \le x_{m+1},\cdots\}$$

with "marginal" distribution

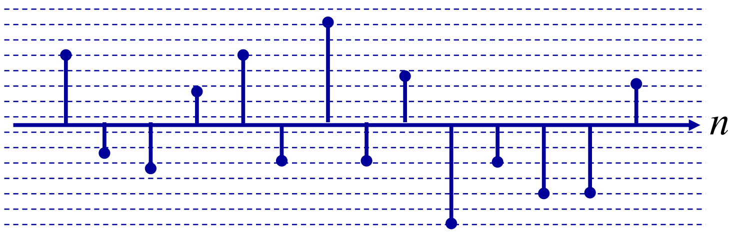
$$P_{\mathbf{x}_m}(x_m, m) = \Pr\{\mathbf{x}_m \le x_m\} \qquad -\infty < m < \infty$$

- Our interest (and ability) in random process is often limited to n^{th} order statistics; e.g., 2^{nd} order statistics, characterized by the following joint distributions

$$P_{\mathbf{x}_k,\mathbf{x}_m}(x_k,k,x_m,m) = \Pr\{\mathbf{x}_k \le x_k \text{ and } \mathbf{x}_m \le x_m\}, \text{ for all } k \text{ and } m$$

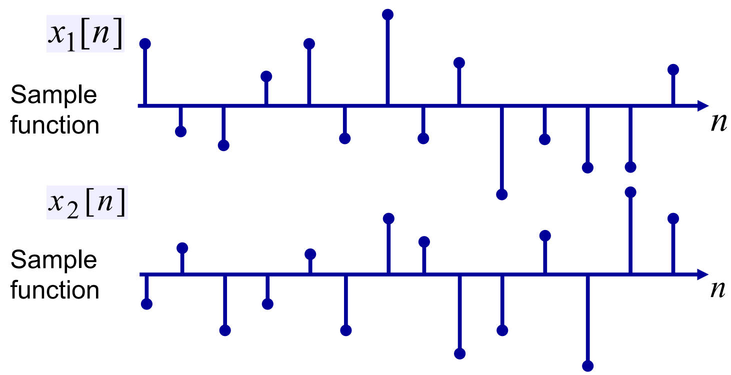
Histogram

- A histogram shows counts of samples that fall in certain "bins". If the boundaries of the bins are close together and we use a sample function with many samples, the histogram provides a good estimate of the probability density function of an (assumed) stationary random process.



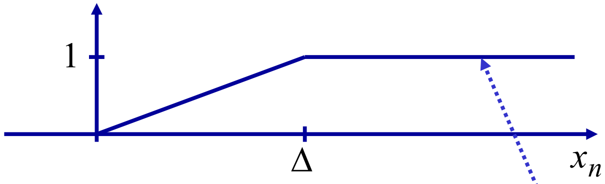
Ensemble of Sample Functions

- We imagine that there are an infinite set of possible sequences where the value at n (i.e., a realized observation) is governed by a probability law. We call this set an ensemble.

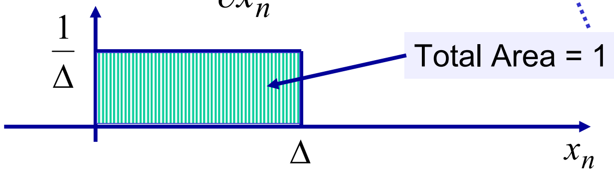


Uniform Distribution

$$P_{\mathbf{x}_n}(x_n,n) = \text{Prob}\{\mathbf{x}_n \le x_n\}$$



$$p_{\mathbf{x}_n}(x_n,n) = \frac{\partial P_{\mathbf{x}_n}(x_n,n)}{\partial x_n}$$

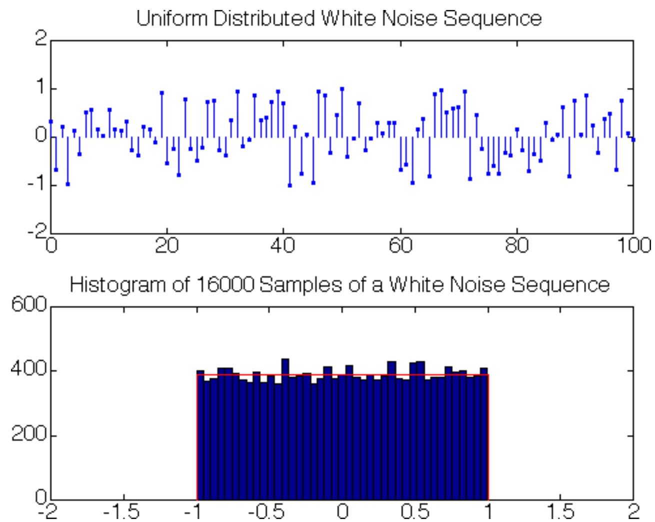


MATLAB Uniform Simulation

- MATLAB's `rand( )` function is useful for such simulations.

```
>> % set Nplt and N
>> x = 2*(rand(1,N)-0.5); %uniform dist.-1 to 1
>> subplot(211); han=stem(0:Nplt-1,x(1:Nplt));
>> set(han,'markersize',3);
>> subplot(212);  hist(x,Nbins); hold on
>> % add theoretical values
>> plot([-1,-1,1,1],N*[0,.5,.5,0]/Nbins,'r');
```


Uniform Random Process



Bernoulli Random Process

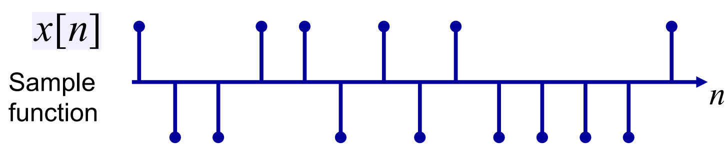
- Suppose that the signal takes on only two different values +1 or -1 with equal probability.

$$P_{\mathbf{x}_n}(x_n,n) = 0.5u(x_n+1) + 0.5u(x_n-1)$$

$$p_{\mathbf{x}_n}(x_n,n) = 0.5\delta(x_n+1) + 0.5\delta(x_n-1)$$

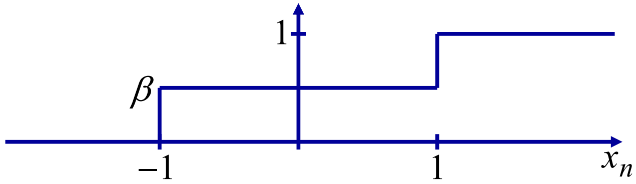
Furthermore, assume that the outcome at time n is independent of all other outcomes.

$$P_{\mathbf{x}_n,\mathbf{x}_m}(x_n,n,x_m,m) = P_{\mathbf{x}_n}(x_n,n)P_{\mathbf{x}_m}(x_m,m)$$

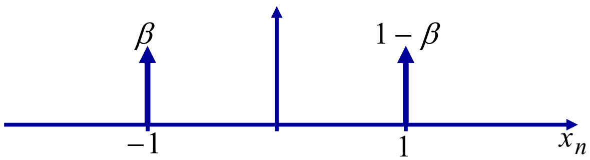


Bernoulli Distribution

$$P_{\mathbf{x}_n}(x_n,n) = \text{Prob}\{\mathbf{x}_n \le x_n\} = \beta u(x+1) + (1-\beta)u(x-1)$$



$$p_{\mathbf{x}_n}(x_n,n) = \frac{\partial P_{\mathbf{x}_n}(x_n,n)}{\partial x_n} = \beta\delta(x+1) + (1-\beta)\delta(x-1)$$

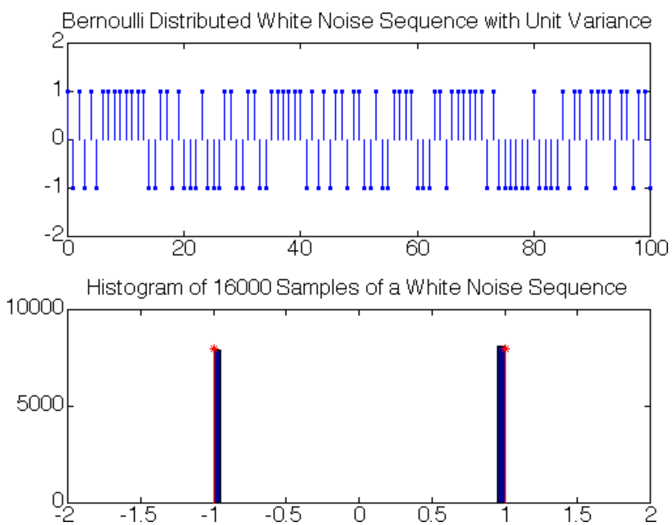


MATLAB Bernoulli Simulation

- MATLAB's `rand( )` function is useful for such simulations.

```
>> % set Nplt and N
>> d = rand(1,N); %uniform dist. Between 0 & 1
>> k = find(d>.5);        %find +1s
>> x = -ones(1,N);        %make vector of all -1s
>> x(k) = ones(1,length(k));   %insert +1s
>> subplot(211); han=stem(0:Nplt-1,x(1:Nplt));
>> set(han,'markersize',3);
>> subplot(212);  hist(x,Nbins); hold on
>> stem([-1,1],N*[.5,.5],'r*'); %add theoretical values
```


Bernoulli Random Process



Averages of Random Processes

- Mean (expected value) of a random process

$$m_{\mathbf{x}_n} = E\{\mathbf{x}_n\} = \int_{-\infty}^{\infty} x p_{\mathbf{x}_n}(x,n)dx$$

- Expected value of a function of a random process

$$E\{g(\mathbf{x}_n)\} = \int_{-\infty}^{\infty} g(x) p_{\mathbf{x}_n}(x,n)dx$$

- In general such averages will depend upon n. However, for a *stationary random process*, all the first-order averages are the same; *e.g.*,

$$m_{\mathbf{x}_n} = m_x \quad \text{for all } n$$

More Averages

- Mean-squared (average power)

$$E\left\{ \mathbf{x}_n \mathbf{x}_n^* \right\} = E\left\{ \left|\mathbf{x}_n\right|^2 \right\} = \int_{-\infty}^{\infty} x^2 p_{\mathbf{x}_n}(x,n)dx$$

- Variance

$$\text{var}[\mathbf{x}_n] = E\left\{ (\mathbf{x}_n - m_{\mathbf{x}_n})(\mathbf{x}_n - m_{\mathbf{x}_n})^* \right\} = \sigma_{\mathbf{x}_n}^2$$

$$\text{var}[\mathbf{x}_n] = E\left\{ \mathbf{x}_n \mathbf{x}_n^* \right\} - \left|m_{\mathbf{x}_n}\right|^2 = \sigma_{\mathbf{x}_n}^2$$

$$\text{var}[\mathbf{x}_n] = \sigma_{\mathbf{x}_n}^2 = \text{mean-square} - (\text{mean})^2$$

Joint Averages of Two R.V.s

- Expected value of a function of two random processes

$$E\{g(\mathbf{x}_n, \mathbf{y}_m)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) p_{\mathbf{x}_n,\mathbf{y}_m}(x,n,y,m)dxdy$$

- Two random processes are *uncorrelated* if

$$E\{\mathbf{x}_n \mathbf{y}_m\} = E\{\mathbf{x}_n\}E\{\mathbf{y}_m\}$$

- Statistical independence implies

$$p_{\mathbf{x}_n,\mathbf{y}_m}(x,n,y,m) = p_{\mathbf{x}_n}(x,n) p_{\mathbf{y}_m}(y,m)$$

- *Independent* random processes are also *uncorrelated*, but not vice versa

Correlation Functions

- Autocorrelation function

$$\phi_{xx}[n,m] = E\left\{ \mathbf{x}_n \mathbf{x}_m^* \right\}$$

- Auto-covariance function

$$\gamma_{xx}[n,m] = E\left\{ (\mathbf{x}_n - m_{\mathbf{x}_n})(\mathbf{x}_m - m_{\mathbf{x}_m})^* \right\}$$

- Cross-correlation function

$$\phi_{xy}[n,m] = E\left\{ \mathbf{x}_n \mathbf{y}_m^* \right\}$$

- Cross-covariance function

$$\gamma_{xy}[n,m] = E\left\{ (\mathbf{x}_n - m_{\mathbf{x}_n})(\mathbf{y}_m - m_{\mathbf{y}_m})^* \right\}$$

Stationary Random Processes

- The probability distributions do not change with time.

$$p_{\mathbf{x}_{n+k}}(x_n, n) = p_{\mathbf{x}_n}(x_n, n)$$

$$p_{\mathbf{x}_{n+k},\mathbf{x}_{m+k}}(x_n, n, x_m, m) = p_{\mathbf{x}_n,\mathbf{x}_m}(x_n, n, x_m, m)$$

- Thus, mean and variance are constant

$$m_x = E\{\mathbf{x}_n\}$$

$$\sigma_x^2 = E\left\{ (\mathbf{x}_n - m_x)(\mathbf{x}_n - m_x)^* \right\}$$

- And the autocorrelation is a one-dimensional function of the time difference

$$\phi_{xx}[n+m,n] = \phi_{xx}[m] = E\left\{ \mathbf{x}_{n+m} \mathbf{x}_n^* \right\}$$

Time Averages

- Time-averages of a random process are random variables themselves.

$$\langle \mathbf{x}_n \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_n$$

$$\langle \mathbf{x}_{n+m}\mathbf{x}_n^* \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_{n+m}\mathbf{x}_n^*$$

- Time averages of a single sample function

$$\langle x[n] \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x[n]$$

$$\langle x[n+m]x^*[n] \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x[n+m]x^*[n]$$

Ergodic Random Processes

- Time-averages are equal to probability averages

$$\langle \mathbf{x}_n \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_n = E\{\mathbf{x}_n\} = m_x$$

$$\langle \mathbf{x}_{n+m}\mathbf{x}_n^* \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_{n+m}\mathbf{x}_n^*$$

$$= E\{\mathbf{x}_{n+m}\mathbf{x}_n^*\} = \phi_{xx}[m]$$

- Estimates from a single sample function

$$\hat{m}_x = \frac{1}{L} \sum_{n=0}^{L-1} x[n] \qquad \hat{\phi}_{xx}[m] = \frac{1}{L} \sum_{n=0}^{L-1} x[n+m]x^*[n]$$

Bernoulli Process Averages

- Mean: $m_x = \int_{-\infty}^{\infty} x[0.5\delta(x+1) + 0.5\delta(x-1)]dx$

$$m_x = \int_{-\infty}^{\infty} 0.5x\delta(x+1)dx + \int_{-\infty}^{\infty} 0.5x\delta(x-1)dx$$

$$m_x = -0.5 + 0.5 = 0$$

- Variance:

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-m_x)^2[0.5\delta(x+1) + 0.5\delta(x-1)]dx$$

$$\sigma_x^2 = 0.5 + 0.5 = 1$$

- Autocorrelation: ($\{\mathbf{x}_n\}$ are assumed independent)

$$\phi_{xx}[m] = \sigma_x^2\delta[m] = \delta[m]$$

Uniform Process Averages

- Mean: $m_x = \int_{-1}^{1} x(0.5)dx = \frac{1}{4}x^2\Big|_{-1}^{1}$

$$m_x = 0.25 - 0.25 = 0$$

- Variance: $\sigma_x^2 = \int_{-1}^{1} (x-m_x)^2(0.5)dx = \frac{1}{6}x^3\Big|_{-1}^{1}$

$$\sigma_x^2 = \tfrac{1}{6} - (-\tfrac{1}{6}) = \tfrac{1}{3}$$

- Autocorrelation: ($\{\mathbf{x}_n\}$ are assumed independent)

$$\phi_{xx}[m] = \sigma_x^2\delta[m] = \frac{1}{3}\delta[m]$$

Properties of the Autocorrelation

- Definition: $\phi_{xx}[m] = E\left\{ x[n+m]x^*[n]\right\}$
- Average power:

$$\phi_{xx}[0] = E\left\{ |x[n]|^2 \right\} = \text{mean} - \text{square}$$

- Symmetry: $\phi_{xx}[-m] = \phi_{xx}^*[m]$

$$\phi_{xx}[-m] = \phi_{xx}[m] \quad \text{if } x \text{ is real}$$

- Shape:

$$|\phi_{xx}[m]| \le \phi_{xx}[0] \qquad \lim_{m\to\infty} \phi_{xx}[m] = |m_x|^2$$

