ECE 4270
Fundamentals of Digital Signal Processing

Lecture 5: Random Signals in LTI Systems

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Georgia Institute of Technology
Summer, 2003

Summary of Averages of Random Processes

- Mean (expected value) of a random process

$$m_{\mathbf{x}_n} = E\{\mathbf{x}_n\} = \int_{-\infty}^{\infty} x p_{\mathbf{x}_n}(x,n)dx$$

- Expected value of a function of a random process

$$E\{g(\mathbf{x}_n)\} = \int_{-\infty}^{\infty} g(x) p_{\mathbf{x}_n}(x,n)dx$$

- In general such averages will depend upon n. However, for a *stationary random process*, all the first-order averages are the same; *e.g.*,

$$m_{\mathbf{x}_n} = m_x \quad \text{for all } n$$

More Averages

- Mean-squared (average power)

$$E\left\{ \mathbf{x}_n \mathbf{x}_n^* \right\} = E\left\{ |\mathbf{x}_n|^2 \right\} = \int_{-\infty}^{\infty} x^2 p_{\mathbf{x}_n}(x,n)dx$$

- Variance

$$\text{var}[\mathbf{x}_n] = E\left\{ (\mathbf{x}_n - m_{\mathbf{x}_n})(\mathbf{x}_n - m_{\mathbf{x}_n})^* \right\} = \sigma_{\mathbf{x}_n}^2$$

$$\text{var}[\mathbf{x}_n] = E\left\{ \mathbf{x}_n \mathbf{x}_n^* \right\} - |m_{\mathbf{x}_n}|^2 = \sigma_{\mathbf{x}_n}^2$$

$$\text{var}[\mathbf{x}_n] = \sigma_{\mathbf{x}_n}^2 = \text{mean-square} - (\text{mean})^2$$

Joint Averages of Two R.V.s

- Expected value of a function of two random processes

$$E\{g(\mathbf{x}_n,\mathbf{y}_m)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) p_{\mathbf{x}_n,\mathbf{y}_m}(x,n,y,m)dxdy$$

- Two random processes are *uncorrelated* if

$$E\{\mathbf{x}_n \mathbf{y}_m\} = E\{\mathbf{x}_n\}E\{\mathbf{y}_m\}$$

- Statistical independence implies

$$p_{\mathbf{x}_n,\mathbf{y}_m}(x,n,y,m) = p_{\mathbf{x}_n}(x,n) p_{\mathbf{y}_m}(y,m)$$

- *Independent* random processes are also *uncorrelated*, but not vice versa

Correlation Functions

- Autocorrelation function

$$\phi_{xx}[n,m] = E\left\{ \mathbf{x}_n \mathbf{x}_m^* \right\}$$

- Auto-covariance function

$$\gamma_{xx}[n,m] = E\left\{ (\mathbf{x}_n - m_{\mathbf{x}_n})(\mathbf{x}_m - m_{\mathbf{x}_m})^* \right\}$$

- Cross-correlation function

$$\phi_{xy}[n,m] = E\left\{ \mathbf{x}_n \mathbf{y}_m^* \right\}$$

- Cross-covariance function

$$\gamma_{xy}[n,m] = E\left\{ (\mathbf{x}_n - m_{\mathbf{x}_n})(\mathbf{y}_m - m_{\mathbf{y}_m})^* \right\}$$

Stationary Random Processes

- The probability distributions do not change with time.

$$p_{\mathbf{x}_{n+k}}(x_n, n) = p_{\mathbf{x}_n}(x_n, n)$$

$$p_{\mathbf{x}_{n+k},\mathbf{x}_{m+k}}(x_n, n, x_m, m) = p_{\mathbf{x}_n,\mathbf{x}_m}(x_n, n, x_m, m)$$

- Thus, mean and variance are constant

$$m_x = E\{\mathbf{x}_n\}$$

$$\sigma_x^2 = E\left\{ (\mathbf{x}_n - m_x)(\mathbf{x}_n - m_x)^* \right\}$$

- And the autocorrelation is a one-dimensional function of the time difference

$$\phi_{xx}[n+m, n] = \phi_{xx}[m] = E\left\{ \mathbf{x}_{n+m} \mathbf{x}_n^* \right\}$$

Time Averages

- Time-averages of a random process are random variables themselves.

$$\langle \mathbf{x}_n \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_n$$

$$\langle \mathbf{x}_{n+m} \mathbf{x}_n^* \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_{n+m} \mathbf{x}_n^*$$

- Time averages of a single sample function

$$\langle x[n] \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x[n]$$

$$\langle x[n+m] x^*[n] \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x[n+m] x^*[n]$$

Ergodic Random Processes

- Time-averages are equal to probability averages

$$\langle \mathbf{x}_n \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_n = E\{\mathbf{x}_n\} = m_x$$

$$\langle \mathbf{x}_{n+m} \mathbf{x}_n^* \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_{n+m} \mathbf{x}_n^*$$

$$= E\left\{ \mathbf{x}_{n+m} \mathbf{x}_n^* \right\} = \phi_{xx}[m]$$

What is $\phi_{xx}[m]$ when $m = 0$?

- Estimates from a single sample function

$$\hat{m}_x = \frac{1}{L} \sum_{n=0}^{L-1} x[n] \qquad \hat{\phi}_{xx}[m] = \frac{1}{L} \sum_{n=0}^{L-1} x[n+m] x^*[n]$$

Bernoulli Process Averages

- Mean: $m_x = \int_{-\infty}^{\infty} x[0.5\delta(x+1) + 0.5\delta(x-1)]dx$

$$m_x = \int_{-\infty}^{\infty} 0.5x\delta(x+1)dx + \int_{-\infty}^{\infty} 0.5x\delta(x-1)dx$$

$$m_x = -0.5 + 0.5 = 0$$

- Variance:

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - m_x)^2[0.5\delta(x+1) + 0.5\delta(x-1)]dx$$

$$\sigma_x^2 = 0.5 + 0.5 = 1$$

- Autocorrelation: ($\{\mathbf{x}_n\}$ are assumed independent)

$$\phi_{xx}[m] = \sigma_x^2\delta[m] = \delta[m]$$

Uniform Process Averages

- Mean: $m_x = \int_{-1}^{1} x(0.5)dx = \frac{1}{4}x^2\Big|_{-1}^{1}$

$$m_x = 0.25 - 0.25 = 0$$

- Variance: $\sigma_x^2 = \int_{-1}^{1} (x - m_x)^2(0.5)dx = \frac{1}{6}x^3\Big|_{-1}^{1}$

$$\sigma_x^2 = \frac{1}{6} - (-\frac{1}{6}) = \frac{1}{3}$$

- Autocorrelation: ($\{\mathbf{x}_n\}$ are assumed independent)

$$\phi_{xx}[m] = \sigma_x^2\delta[m] = \frac{1}{3}\delta[m]$$

Properties of the Autocorrelation

- Definition: $\phi_{xx}[m] = E\{x[n+m]x^*[n]\}$
- Average power:

$$\phi_{xx}[0] = E\{|x[n]|^2\} = \text{mean} - \text{square}$$

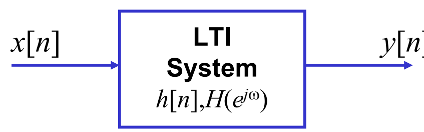
- Symmetry: $\phi_{xx}[-m] = \phi_{xx}^*[m]$

$$\phi_{xx}[-m] = \phi_{xx}[m] \quad \text{if } x \text{ is real}$$

- Shape:

$$|\phi_{xx}[m]| \le \phi_{xx}[0] \qquad \lim_{m\to\infty} \phi_{xx}[m] = |m_x|^2$$

Effect of a Linear System on the Mean of a Random Input

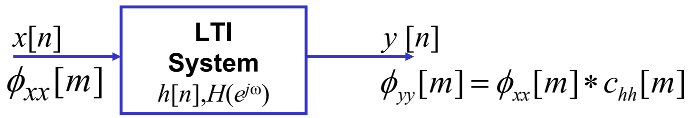


$$m_y = E\left\{\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right\} = \sum_{k=-\infty}^{\infty} E\{x[k]\}h[n-k]$$

$$m_y = E\{x[k]\}\left(\sum_{k=-\infty}^{\infty} h[n-k]\right) = m_x\left(\sum_{k=-\infty}^{\infty} h[k]\right)$$

$$m_y = m_x\left(\sum_{k=-\infty}^{\infty} h[k]\right) = m_x H(e^{j0})$$

Effect of a Linear System on the Autocorrelation of a Random Input



$$\phi_{yy}[m] = E\left\{ y[n+m]y^*[n] \right\}$$ Autocorrelation of output

$$= E\left\{ \sum_{r=-\infty}^{\infty} x[n+m-r]h[r] \sum_{k=-\infty}^{\infty} x^*[n-k]h^*[k] \right\}$$

$$\phi_{yy}[m] = \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]c_{hh}[\ell]$$

Deterministic autocorrelation of impulse response

$$c_{hh}[m] = \sum_{k=-\infty}^{\infty} h[m+k]h^*[k]$$

Computing Average Power Output

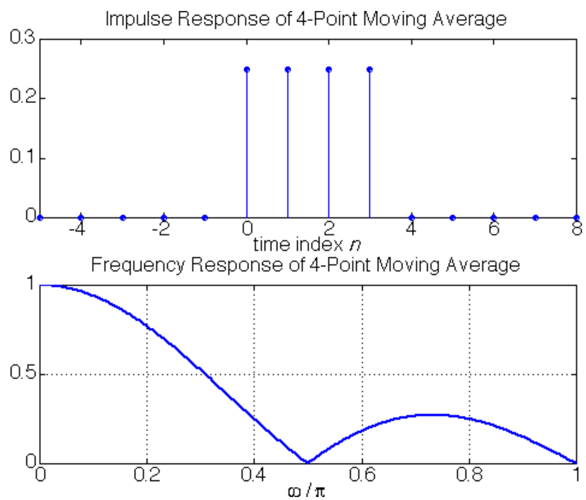
- Assume a zero mean input whose average power is σ_x^2 and autocorrelation function is $\phi_{xx}[m] = \sigma_x^2\delta[m]$.
- The autocorrelation of the output is

$$\phi_{yy}[m] = \sigma_x^2\delta[m] * c_{hh}[m] = \sigma_x^2 c_{hh}[m]$$

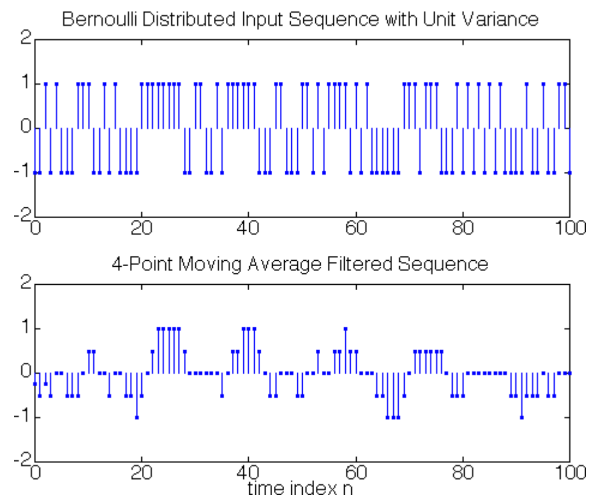
- The average power of the output is easily found from this result as

$$\sigma_y^2 = \phi_{yy}[0] = \sigma_x^2 c_{hh}[0] = \sigma_x^2 \sum_{k=-\infty}^{\infty} h[k]h^*[k]$$

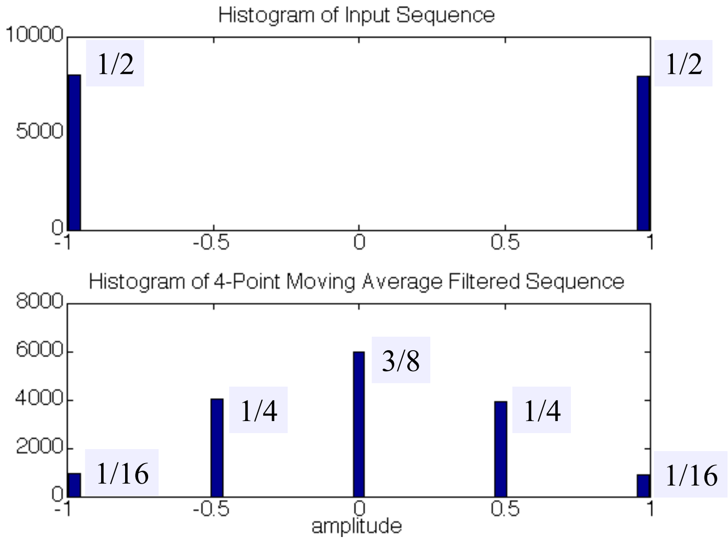
MATLAB Experiment - I



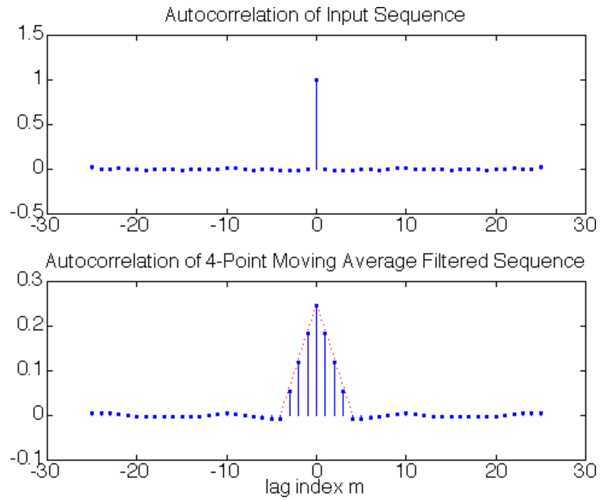
MATLAB Experiment - II



MATLAB Experiment - III



MATLAB Experiment - IV



A Probability Theorem

- The probability density function for a sum of *M* identically distributed independent random variables is equal to the (*M*-1)-fold convolution of the probability density function with itself; *e.g, M*=2,

$$p_{\mathbf{y}}(y) = p_{\mathbf{x}}(y) * p_{\mathbf{x}}(y) = \int_{-\infty}^{\infty} p_{\mathbf{x}}(y-x) p_{\mathbf{x}}(x) dx$$

- Example: Suppose $y[n]=x[n]+x[n-1]$. Then if

$$p_{\mathbf{x}}(x) = 0.5\delta(x+1) + 0.5\delta(x-1)$$

it follows that

$$p_{\mathbf{y}}(y) = 0.25\delta(y+2) + 0.5\delta(y) + 0.25\delta(y-2)$$

Central Limit Theorem

- The probability density of the sum of a large number of independent random variables approaches a Gaussian distribution.

$$p_{\mathbf{y}}(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-(y-m_y)^2/2\sigma_y^2}$$

- Since filters perform a weighted sum of the samples of the input, the output of a digital filter for a random input tends to have a Gaussian distribution.

Power Density Spectrum

- The concept of power of a random signal at a particular frequency – it can only be estimated.
- The power density spectrum of a random signal is the DTFT of the autocorrelation function, which is an expectation, $E(\bullet)$.

$$\Phi(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m]\, e^{-j\omega m}$$

$$\phi_{xx}[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{j\omega}) e^{j\omega m}\, d\omega$$

$$E\left\{ |x|^2 \right\} = \phi_{xx}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{j\omega})\, d\omega$$

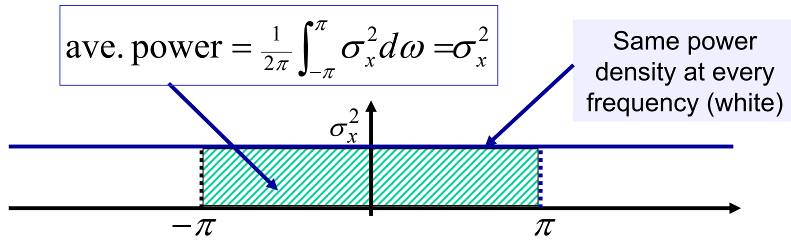
White Noise

- Consider a zero mean signal whose autocorrelation function is

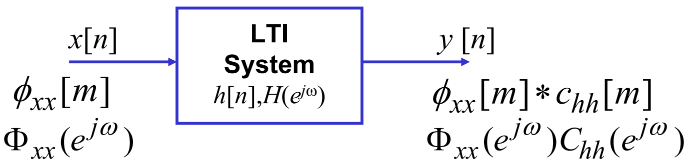
$$\phi_{xx}[m] = \sigma_x^2 \delta[m]$$

- The power spectrum of this signal is

$$\Phi_{xx}[m] = \sigma_x^2 \quad |\omega| \le \pi$$



Effect of a Linear System on the Power Spectrum of a Random Input



$$\phi_{yy}[m] = \phi_{xx}[m] * c_{hh}[m]$$

$$c_{hh}[m] = \sum_{k=-\infty}^{\infty} h[m+k] h^*[k] = h[-m] * h^*[m]$$

$$\Phi_{yy}(e^{j\omega}) = \Phi_{xx}(e^{j\omega}) C_{hh}(e^{j\omega})$$

$$C_{hh}(e^{j\omega}) = H(e^{-j\omega}) H^*(e^{-j\omega}) = \left|H(e^{-j\omega})\right|^2$$

Properties of Power Density

- Real: $\Phi^*(e^{j\omega}) = \Phi(e^{j\omega})$
- Symmetry:

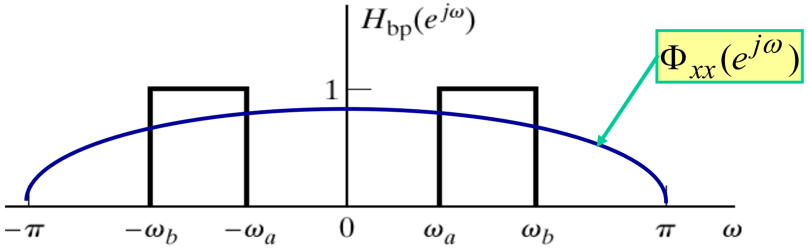
$$\Phi(e^{j\omega}) = \Phi(e^{-j\omega}) \quad \text{if } x[n] \text{ is real}$$

- Positivity: $\Phi(e^{j\omega}) \ge 0$
- Magnitude-squared has same properties:

$$C_{hh}^*(e^{j\omega}) = \left|H(e^{-j\omega})\right|^2 = C_{hh}(e^{j\omega}) \quad \Rightarrow \quad \text{real}$$

$$C_{hh}(e^{-j\omega}) = \left|H(e^{j\omega})\right|^2 = C_{hh}(e^{j\omega}) \quad \text{if } x[n] \text{ is real}$$

Example - Bandpass Filter

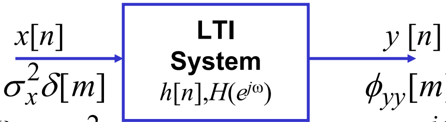


$$E\left\{|y[n]|^2\right\} = \phi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})\left|H_{bp}(e^{j\omega})\right|^2 d\omega$$

$$E\left\{|y[n]|^2\right\} = \frac{1}{2\pi}\int_{\omega_a}^{\omega_b} \Phi_{xx}(e^{j\omega})d\omega + \frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} \Phi_{xx}(e^{j\omega})d\omega \geq 0$$

$$\Rightarrow \quad \Phi_{xx}(e^{j\omega}) \geq 0$$

Linear System with a White Noise Input



$\phi_{xx}[m] = \sigma_x^2 \delta[m]$

$\Phi_{xx}(e^{j\omega}) = \sigma_x^2$

$\phi_{yy}[m] = \sigma_x^2 c_{hh}[m]$

$\Phi_{yy}(e^{j\omega}) = \sigma_x^2 C_{hh}(e^{j\omega})$

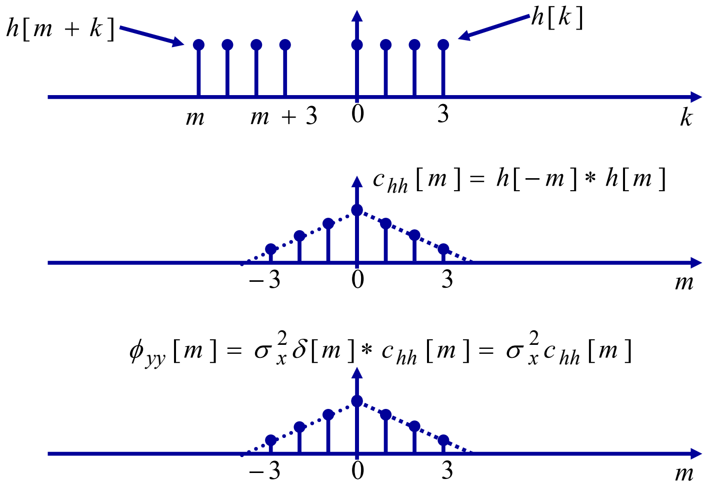
$$\phi_{yy}[m] = \sigma_x^2 \delta[m] * c_{hh}[m] = \sigma_x^2 c_{hh}[m]$$

$$c_{hh}[m] = \sum_{k=-\infty}^{\infty} h[m+k]h^*[k] = h[-m] * h^*[m]$$

$$\Phi_{yy}(e^{j\omega}) = \Phi_{xx}(e^{j\omega})C_{hh}(e^{j\omega}) = \sigma_x^2 C_{hh}(e^{j\omega})$$

$$C_{hh}(e^{j\omega}) = H(e^{-j\omega})H^*(e^{-j\omega}) = \left|H(e^{-j\omega})\right|^2$$

White Noise into Moving Average



White Noise into Moving Average

- Frequency response of filter:

$$H(e^{j\omega}) = \frac{1}{M+1}\sum_{n=0}^{M} e^{-j\omega n} = \frac{\sin[(M+1)\omega/2]}{(M+1)\sin(\omega/2)} e^{-j\omega M/2}$$

- Power spectrum of output when the input is white noise:

$$\Phi_{yy}(e^{j\omega}) = \sigma_x^2 \left|H(e^{j\omega})\right|^2$$

$$= \sigma_x^2 \left(\frac{\sin[(M+1)\omega/2]}{(M+1)\sin(\omega/2)}\right)^2$$

4-Point Moving Average Filter