ECE 4270
Fundamentals of Digital Signal Processing

Lecture 6: The z Transform

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The Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Fourier transform is an infinite sum and does not converge for all sequences.
- Complex sinusoid $e^{j\omega}$ is a particular complex variable – recall eigenfunctions of LTI systems.
- It may be more convenient to address analytical problems in terms of a general complex variable, z.

Complex Derivative & Cauchy-Riemann Eqs.

$$f(x,y) \equiv u(x,y) + jv(x,y) \qquad z \equiv x + jy \qquad dz = dx + jdy$$

$$\frac{\partial x}{\partial z} = 1 \qquad \frac{\partial y}{\partial z} = -j \qquad \frac{df}{dz} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y}$$

$$\frac{df}{dz} = \left(\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}\right) - j\left(\frac{\partial u}{\partial y} + j\frac{\partial v}{\partial y}\right) = \left(\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}\right) + \left(-j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$$

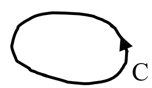
If f is **complex differentiable**, then the value of the derivative must be the same for a given dz, regardless of its orientation (e.g., along x- or y-axis or any other direction). Thus, it requires that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Cauchy Integral

- **Cauchy Integral Theorem**

  If $f(z)$ is analytic in some simply connected region or domain D, then for any closed contour C completely contained in D

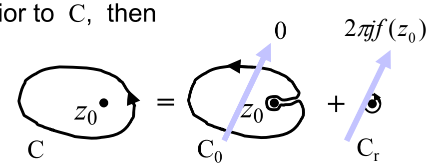
$$\oint_C f(z)dz = 0$$



- **Cauchy Integral Formula**

  Let $f(z)$ be analytic in the simply connected domain D, and let C be a simple closed positively oriented contour that lies in D. If z_0 is a point that lies interior to C, then

$$f(z_0) = \frac{1}{2\pi j}\oint_C \frac{f(z)}{z - z_0}dz$$



The z-Transform

- The z -transform of a sequence is defined as

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Since this is generally an infinite sum, we need to be concerned about "convergence"; *i.e.*, is the sum finite? In general, convergence will depend upon z; *e.g.*, $0 \le r_R < |z| < r_L < \infty$.
- The inverse z-transform is given by the contour integral

$$x[n]=\frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Region of Convergence

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$|X(z)|=\left|\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n}$$

- The region of convergence is the set of values of z such that $0 \le r_R < |z| < r_L < \infty$

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

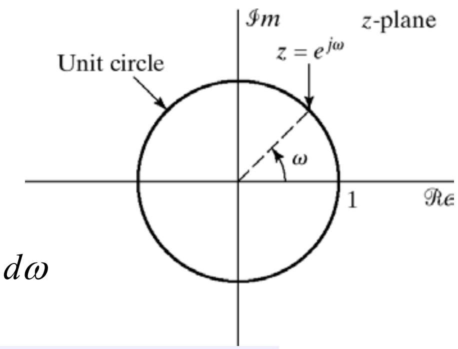
$|z|^{-n}$ can "tame" a growing sequence if z is properly chosen

Relation to DTFT

- The DTFT is equal to the z-transform evaluated on the unit circle:

$$X(z)|_{z=e^{j\omega}}=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= X(e^{j\omega}) \;=\; \text{DTFT}$$



$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \Rightarrow \text{ ROC contains } |z|=1$$

Remember These Formulas

- A tricky manipulation:

$$a_0 + a_1 z^{-1} + a_0 z^{-2} = z^{-1}\left(a_0 z + a_1 + a_0 z^{-1}\right)$$
$$= z^{-1}\left(a_1 + a_0(z+z^{-1})\right)$$

- Summing a finite exponential sequence:

$$\sum_{n=N_1}^{N_2} \alpha^n = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha} \qquad N_2 \ge N_1$$

- Summing an infinite exponential sequence:

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad \text{if } |\alpha| < 1$$

Examples

- Impulse sequence:

$$x[n] = \delta[n - n_0]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] z^{-n} = z^{-n_0}$$

- Pulse sequence:

$$x[n] = \begin{cases} 1 & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{M-1} z^{-n} = \frac{1 - z^{-M}}{1 - z^{-1}} \quad z \neq 0$$

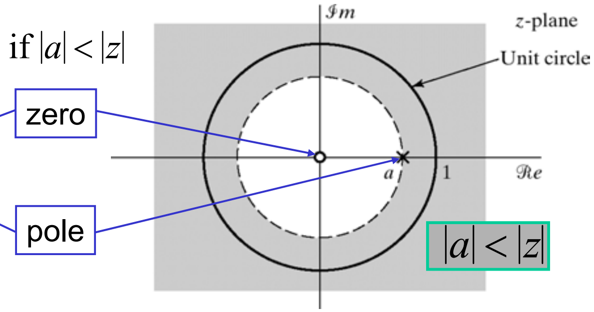
Right-Sided Exponential Signal

- Right-sided exponential sequence: $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-(az^{-1})} \quad \text{if } \left|az^{-1}\right| < 1$$

$$X(z) = \frac{1}{1 - az^{-1}} \quad \text{if } |a| < |z|$$

$$= \frac{(z-0)}{(z-a)}$$



Left-Side Exponential Signal

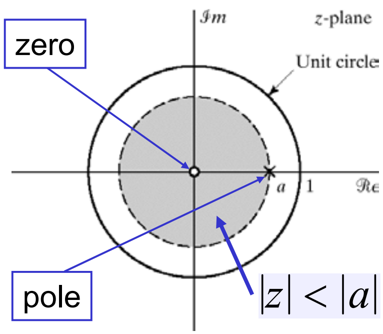
- Left-sided exponential sequence: $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{1 - a^{-1}z} \quad \text{if } \left|a^{-1}z\right| < 1$$

$$= \frac{1 - a^{-1}z - 1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}$$



Two-Sided Exponential Signal

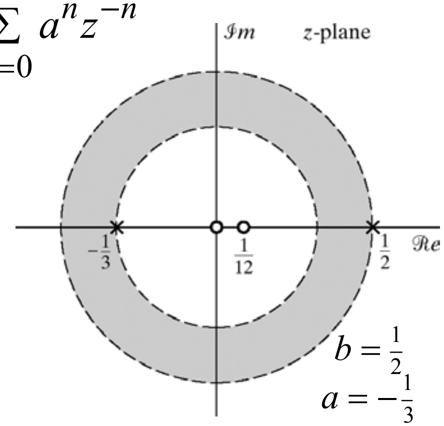
$$x[n] = -b^n u[-n-1] + a^n u[n]$$

$$X(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

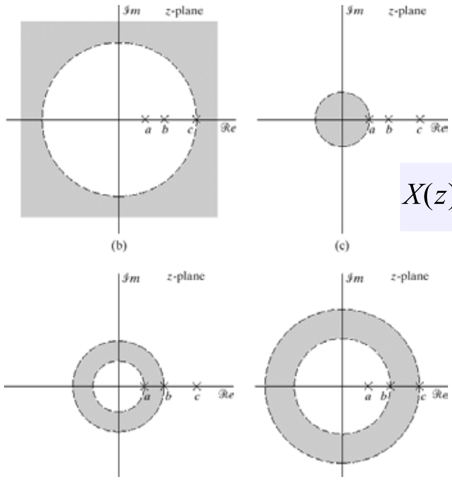
$$= \underset{\text{if } |z|<|b|}{\frac{1}{1-bz^{-1}}} + \underset{\text{if } |z|>|a|}{\frac{1}{1-az^{-1}}}$$

$$= \frac{2-(a+b)z^{-1}}{(1-az^{-1})(1-bz^{-1})}$$

if $|a|<|z|<|b|$



Which ROC?



$$X(z) = \frac{A}{1 - az^{-1}} + \frac{B}{1 - bz^{-1}} + \frac{C}{1 - cz^{-1}}$$

- The sequence that corresponds to this z Transform depends on the ROC for which it is valid!

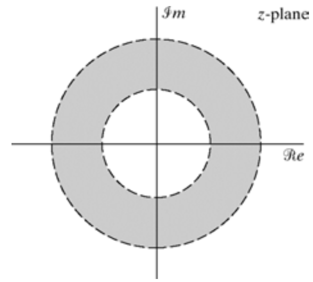
Properties of the z-Transform - I

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.



Properties of the z-Transform - II

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

Transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1 - z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1 - az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\|z\| > \|a\|$

These are key to partial fraction expansion method

Transform Pairs

Sequence	Transform	ROC
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1}\cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1}\cap R_{x_2}$
10	3.4.8	Initial-value theorem: $x[n]=0,\quad n<0$ $\quad\lim_{z\to\infty} X(z)=x[0]$		

But first a few more words on the z Transform Region of Convergence …

The z Transform of $u[n]$

- Our standard approach:

$$X(z)=\sum_{n=-\infty}^{\infty}u[n]\,z^{-n}=\sum_{n=0}^{\infty}z^{-n}=\sum_{n=0}^{\infty}\left(z^{-1}\right)^n$$

$$=\frac{1}{1-z^{-1}}\ \text{provided}\ \left|z^{-1}\right|<1\Rightarrow|z|>1$$

- We come to same conclusion about the ROC from our absolute summability criterion:

$$|X(z)|=\left|\sum_{n=0}^{\infty}z^{-n}\right|\le\sum_{n=0}^{\infty}\left|z^{-1}\right|^n\Rightarrow|z|>1$$

$X(z)$ for $|z| = 1$

- $X(z)$ has a pole ("blows up") at $z = 1$:

$$X(z)\Big|_{z=1} = \frac{1}{1-z^{-1}}\Bigg|_{z=1} = \frac{1}{0} \to \infty$$

- But what about $X(z)$ for other values of z where $|z| = 1$, but $z \neq 1+j0 = 1$? Example: $z = +j$:

$$X(z)\Big|_{z=+j} = \frac{1}{1-z^{-1}}\Bigg|_{z=+j} = \frac{1}{1+j}$$

- This result seems well defined, but is it valid? Does the series definition of $X(z)$ really converge to this?

$X(z)$ for $|z| = 1$, Continued

- Back to the definition:

$$X(z)\Big|_{z=j} = \sum_{n=0}^{\infty}\left(j^{-1}\right)^n = \sum_{n=0}^{\infty}\left(-j\right)^n$$

$$= 1 - j - 1 + j + 1 - \ldots \neq \frac{1}{1+j}$$

- So this was an example where $|z| = 1$, $z \neq 1$; $X(z)$ doesn't "blow up", but it doesn't converge either.

$X(z)$ for $|z| = 1$, Concluded

- Consider:

$$X_N(z) \equiv \sum_{n=-N}^{N} u[n]\, z^{-n} = \sum_{n=0}^{N} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}};$$

$$\lim_{N\to\infty} X_N(z) = X(z)$$

- Now consider a general complex $z = |z|\, e^{j\theta_z}$:

$$X_N(z) = \frac{1-|z|^{-N} e^{-jN\theta_z}}{1-|z|^{-1} e^{-j\theta_z}};\ \lim_{N\to\infty} X_N(z) = X(z)$$

- If $|z| = 1$,

$$\lim_{N\to\infty} X_N(z) = \lim_{N\to\infty} \frac{1-e^{-jN\theta_z}}{1-e^{-j\theta_z}}$$

- Doesn't "blow up" unless $\theta_z = 0$, but doesn't converge for any z with $|z| = 1$!

Final Comment

- For this one case, we have shown that
  - $X(z)$ is infinite ("blows up") only at $z = 1$, and
  - $X(z)$ is finite but fails to converge to a fixed value for all other z having $|z| = 1$
- So in this case, convergence really does require $|z| > 1$, not just $z \neq 1$
- Is this true in general?