ECE 4270

Fundamentals of Digital Signal Processing

Lecture 7: Inverse z Transforms

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4 Ways to Invert the z-Transform

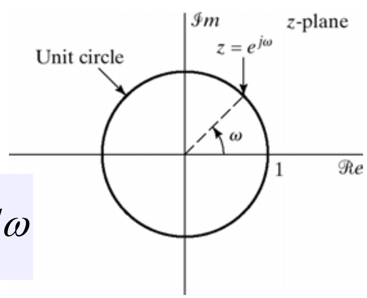
- By the contour integral of the inverse transform definition
- "By Inspection" - recognize common transform pairs
- Partial Fraction Expansion - effective for rational *z*-transforms
- Power Series Expansion

Inversion by Contour Integration

Recall the Cauchy integral definition of the inverse z transform:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

The inverse DTFT is an example of this for the case $z = e^{j\omega}$, which also implies the contour *C* is chosen as the unit circle:



$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

We'll leave this to course on complex variables ….

Concept of the Partial Fraction Expansion Inversion Method - 1

- Consider a general rational z-transform (no repeated roots); two equivalent forms are:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Concept of the Partial Fraction Expansion Inversion Method - 2

We can always find a partial fraction expansion in the third equivalent form

$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{only if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

And then invert by inspection:

4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$

Doing the Partial Fraction Expansion - 1

To find the coefficients of the m^{th} first-order pole, consider

$$(1 - d_m z^{-1})X(z) = (1 - d_m z^{-1})\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right] + \sum_{k=1}^{N} \frac{(1 - d_m z^{-1})A_k}{1 - d_k z^{-1}}$$

Now evaluate at $z = d_m$ …

Doing the Partial Fraction Expansion - 2

Thus we now have:

$$(1 - d_m z^{-1})X(z)\Big|_{z=d_m} = (1 - \overset{0}{\cancel{d_m z^{-1}}})\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]_{z=d_m} + \sum_{k=1}^{N} \frac{(1 - \overset{0}{\cancel{d_m z^{-1}}})A_k}{1 - d_k z^{-1}}\Bigg|_{z=d_m} = A_m$$

$$A_k = (1 - d_k z^{-1})X(z)\Big|_{z=d_k}$$

We will see that this formula is applied to the product form in practice …

Writing Down $x[n]$

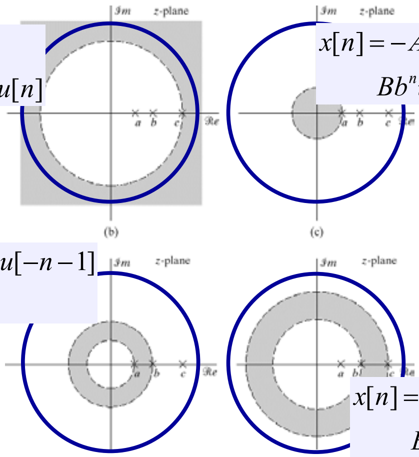
$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

(We'll work out the long division by example in a minute …)

$$x[n] = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r \delta[n-r]\right]}_{\text{if } M \geq N} + \underbrace{\sum_{k} A_k d_k^n u[n]}_{\text{when } |d_k| < r_R} - \underbrace{\sum_{k} A_k d_k^n u[-n-1]}_{\text{when } |d_k| > r_L}$$

$x[n]$ Depends on Knowing the ROC

$$X(z) = \frac{A}{1-az^{-1}} + \frac{B}{1-bz^{-1}} + \frac{C}{1-cz^{-1}}$$



$$x[n] = Aa^n u[n] + Bb^n u[n] + Cc^n u[n]$$

$$x[n] = -Aa^n u[-n-1] - Bb^n u[-n-1] - Cc^n u[-n-1]$$

$$x[n] = Aa^n u[n] - Bb^n u[-n-1] - Cc^n u[-n-1]$$

$$x[n] = Aa^n u[n] + Bb^n u[n] - Ca^n u[-n-1]$$

Partial Fraction Expansion Example

$$X(z) = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+2z^{-2}} = \frac{(1+z^{-1})^3}{(1-z^{-1})(1-2z^{-1})}$$

Long Division

$$X(z) = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+2z^{-2}} = \frac{(1+z^{-1})^3}{(1-z^{-1})(1-2z^{-1})}$$

$$\begin{array}{r|l}
 & 0.5z^{-1} + 2.25 \\
\hline
2z^{-2} - 3z^{-1} + 1 & z^{-3} + 3.0z^{-2} + 3.0z^{-1} + 1.0 \\
 & z^{-3} - 1.5z^{-2} + 0.5z^{-1} \\
 & \quad 4.5z^{-2} + 2.50z^{-1} \\
 & \quad 4.5z^{-2} - 6.75z^{-1} + 2.25 \\
 & \qquad 9.25z^{-1} - 1.25
\end{array}$$

$$X(z) = 2.25 + 0.5z^{-1} - \frac{1.25 - 9.25z^{-1}}{1-3z^{-1}+2z^{-2}} = 2.25 + 0.5z^{-1} - \frac{1.25-9.25z^{-1}}{(1-z^{-1})(1-2z^{-1})}$$

Finding the Coefficients of the Poles

$$X(z) = \frac{(1+z^{-1})^3}{(1-z^{-1})(1-2z^{-1})}$$

$$= 2.25 + 0.5z^{-1} + \frac{A_1}{(1-z^{-1})} + \frac{A_2}{(1-2z^{-1})}$$

$$A_1 = X(z)(1-z^{-1})\Big|_{z=1} = \frac{(1+z^{-1})^3}{(1-2z^{-1})}\Bigg|_{z=1} = \frac{8}{-1} = -8$$

$$A_2 = X(z)(1-2z^{-1})\Big|_{z=2} = \frac{(1+z^{-1})^3}{(1-z^{-1})}\Bigg|_{z=2} = \frac{(3/2)^3}{1/2} = 6.75$$

$$X(z) = 2.25 + 0.5z^{-1} + \frac{-8}{(1-z^{-1})} + \frac{6.75}{(1-2z^{-1})}$$

Writing Down $x[n]$

$$X(z) = 2.25 + 0.5z^{-1} + \frac{-8}{(1-z^{-1})} + \frac{6.75}{(1-2z^{-1})}$$

- If ROC is $2 < |z|$

$$x[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] + 6.75(2)^n u[n]$$

- If ROC is $1 < |z| < 2$

$$x[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] - 6.75(2)^n u[-n-1]$$

- If ROC is $|z| < 1$

$$x[n] = 2.25\delta[n] + 0.5\delta[n-1] + 8u[-n-1] - 6.75(2)^n u[-n-1]$$

Repeated Roots

- See text for extension of the technique to repeated roots:

$$X(z) = \frac{(1+z^{-1})^3}{(1-z^{-1})^2(1-2z^{-1})}$$

double pole at $z = 1$

Partial Fraction Expansion in MATLAB

$$X(z) = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+2z^{-2}} = 2.25 + 0.5z^{-1} + \frac{-8}{1-z^{-1}} + \frac{6.75}{1-2z^{-1}}$$

```
» [r,p,k]=residuez([1,3,3,1],[1,-3,2])
r =
   6.75000000000000
  -8.00000000000000
p =
     2
     1
k =
   2.25000000000000   0.50000000000000
```

- *MATLAB's* **`residuez`** *can also go back the other way, and can also handle repeated roots*
  - *Note: '***`residuez`***' is part of the optional Signal Processing Toolbox*

How Rational z Transforms Arise in the Analysis of LTI Systems

Selected z-Transform Theorems

- The delay or shift property:

$$x[n-n_0] \quad \Leftrightarrow \quad z^{-n_0}X(z)$$

- The convolution property:

$$y[n]=x[n]*h[n] \quad \Leftrightarrow \quad Y(z)=H(z)X(z)$$

- A consequence:

$$y[n]=x[n]*\delta[n-n_0] \quad \Leftrightarrow \quad Y(z)=z^{-n_0}X(z)$$

An IIR System

- Difference equation:

$$y[n]=ay[n-1]+x[n] \Leftrightarrow Y(z)=az^{-1}Y(z)+X(z)$$

- System function:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1-az^{-1}}=\frac{z}{z-a} \qquad |z|>|a|$$

ROC if causal

- Frequency response:

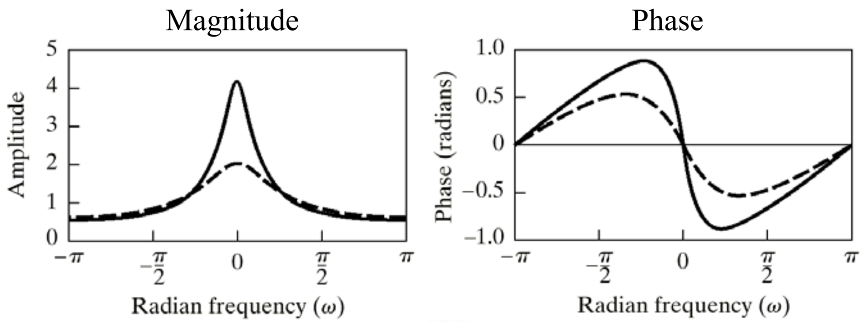
$$H(e^{j\omega})=H(z)\big|_{z=e^{j\omega}}=\frac{1}{1-ae^{-j\omega}}$$

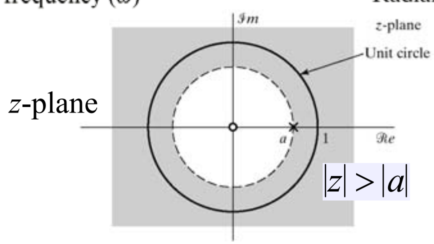
Assumes $|a|<1$

- Impulse response:

$$h[n]=a^n u[n]$$

IIR Frequency Response





System Function of a Difference Equation (DE)

$$\sum_{k=0}^{N} a_k y[n-k]=\sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k Y(z)z^{-k}=\sum_{k=0}^{M} b_k X(z)z^{-k}$$

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right)Y(z)=\left(\sum_{k=0}^{M} b_k z^{-k}\right)X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}{\left(\sum_{k=0}^{N} a_k z^{-k}\right)}$$

Difference equations give rise to rational z transforms!

$H(z)$ and $h[n]$

- Consider a causal system; *i.e.* $h[n]=0$ for $n<0$:

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

ROC: $|z|>\max_k |d_k|$

$$H(z)=\underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{only if } M\geq N}+\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

$$h[n]=\sum_{r=0}^{(M-N)} B_r\delta[n-r]+\sum_{k=1}^{N} A_k d_k^n u[n]$$

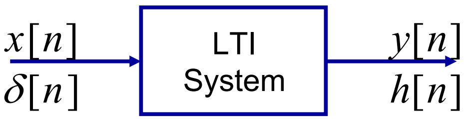
Frequency Response of a DE

$$\sum_{k=0}^{N} a_k y[n-k]=\sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\omega})=H(z)\big|_{z=e^{j\omega}}=\frac{\left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right)}{\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right)}$$

ROC must Contain the Unit circle

LTI System Characterizations



$x[n]$	$y[n]$
$\delta[n]$	$h[n]$
$e^{j\omega n}$	$H(e^{j\omega})e^{j\omega n}$
$X(e^{j\omega})$	$X(e^{j\omega})H(e^{j\omega})$
$X(z)$	$X(z)H(z)$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}{\left(\sum_{k=0}^{N} a_k z^{-k}\right)}$$

$$H(e^{j\omega})=H(z)\big|_{z=e^{j\omega}}$$