ECE 4270

Fundamentals of Digital Signal Processing

Lecture 8: Sampling

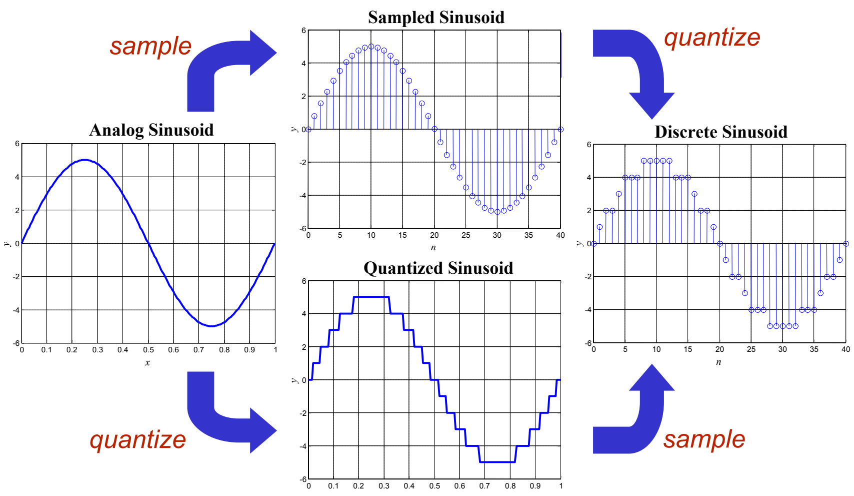
School of Electrical and Computer Engineering
Georgia Institute of Technology

Summer, 2004

Discrete Signals

- Discrete in time
  - Needs **sampling**
  - Issue: Under what condition, the original continuous-time signal can be identically recovered or reproduced?
- Discrete in value
  - Needs **quantization**
  - Issue: Deviation from original continuous-valued signal is inevitable, but how to minimize this deviation or distortion?
- Discrete in time and value
  - Our practical, computational interests

Discrete Signals

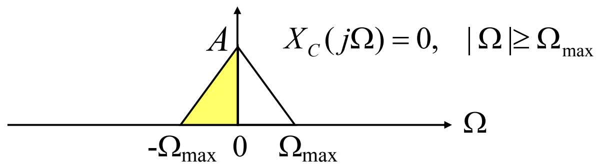


Digital Processing of Analog Signals

- **A-to-D conversion:** sampling *and* quantization
- **Numerical algorithm:** convolution, difference equations, DFT, LPC
  - Implemented on DSP chips, computers or ASICs with finite-precision arithmetic
- **D-to-A conversion:** quantization *and* filtering (why?)

Bandlimited Signals

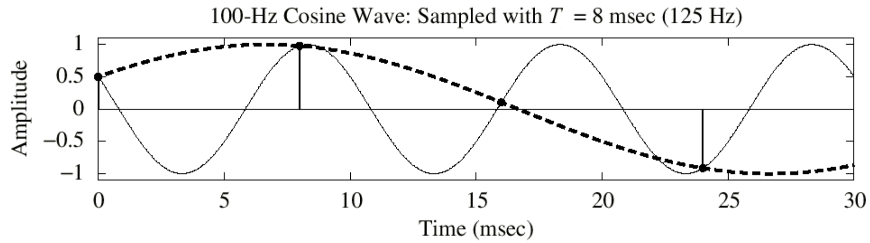
- A bandlimited continuous-time (analog) signal has a Fourier transform that is zero for all frequencies above some highest frequency, say Ω_{max}.



- Simple example is a sinusoidal signal

$$x_c(t) = A\cos(\Omega_0 t + \phi) \qquad \Omega_{max} = \Omega_0 + \varepsilon$$

The Sampling Theorem



A bandlimited signal with highest frequency component at Ω_{max} can be reconstructed exactly from samples taken with sampling frequency Ω_s (radians/s)

$$\Omega_s = \frac{2\pi}{T} > 2\Omega_{max}$$

Sampling a Sinusoidal Signal

$$x_1(t) = A\cos(\Omega_0 t + \phi)$$

$$x_1[n] = A\cos(\Omega_0 nT + \phi) = A\cos(\omega_0 n + \phi) \qquad \omega_0 = \Omega_0 T$$

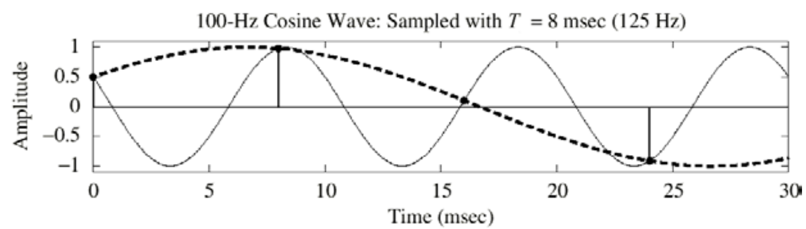
$$x_2(t) = A\cos[(\Omega_0 + 2\pi/T)t + \phi]$$

$$\begin{aligned} x_2[n] &= A\cos[(\Omega_0 + 2\pi/T)nT + \phi)] \\ &= A\cos(\Omega_0 nT + 2\pi n + \phi) = A\cos(\Omega_0 nT + \phi) \end{aligned}$$

$$x_3(t) = A\cos[(2\pi/T - \Omega_0)t - \phi]$$

$$\begin{aligned} x_3[n] &= A\cos[(2\pi/T - \Omega_0)nT - \phi)] \\ &= A\cos(2\pi n - \Omega_0 nT - \phi) = A\cos(\Omega_0 nT + \phi) \end{aligned}$$

Aliasing and Reconstruction



- Each of the sinusoidal components of a signal has an infinite number of aliases – recall that DTFT is periodic with period 2π.
- ***Nyquist Frequency*** Ω_N is the highest frequency that a sampled signal can unambiguously represent and is equal to half the sampling rate, $\Omega_s/2$. ***Nyquist Rate*** is defined as $2\Omega_{max}$, the lowest sampling rate to avoid aliasing.
- The ideal D-to-C converter reconstructs the lowest frequency alias of each sinusoidal component. This is why we need to constrain our sampling rate so that $\Omega_s > 2\Omega_{max}$

Sampling (C-to-D Conversion)

- Discrete-time Fourier transform:

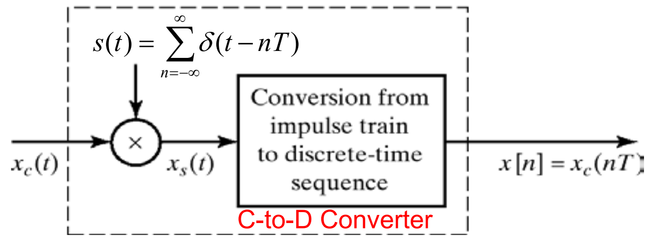
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Frequency-domain relation:

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega T n} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

- Sampling frequency: $\Omega_s = 2\pi/T$
- Normalized frequency: $\omega = \Omega T$

Derivation of Basic FT Formula - I



$$x_s(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

Derivation of Basic FT Formula - II

$$x_s(t) = x_c(t) \cdot s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \Leftrightarrow S(j\Omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta(\Omega - k\frac{2\pi}{T})$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta(\Omega - k\frac{2\pi}{T})$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(\Omega - k\frac{2\pi}{T})$$

Derivation of Basic FT Formula - III

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

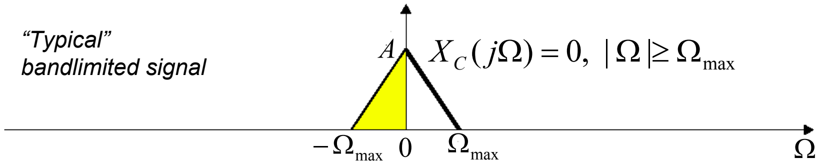
$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT}$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega T)n} = X(e^{j\Omega T})$$

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(\Omega - k\frac{2\pi}{T})$$

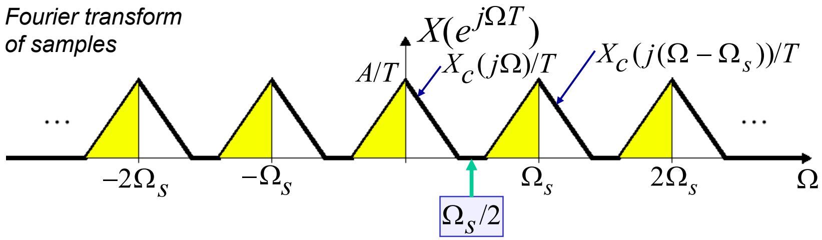
Oversampling

“Typical” bandlimited signal



$$X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \qquad \Omega_s = 2\pi/T$$

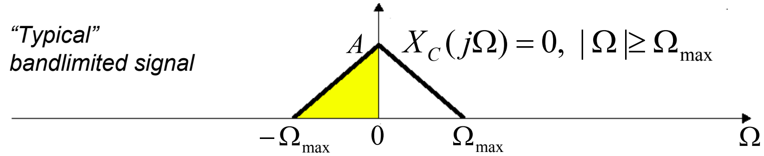
Fourier transform of samples



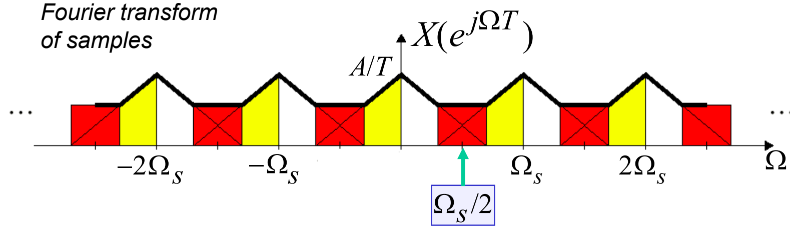
Undersampling (Aliasing Distortion)

- If $\Omega_s < 2\Omega_{\max}$, the copies of $X_c(j\Omega)$ overlap, and we have aliasing distortion.

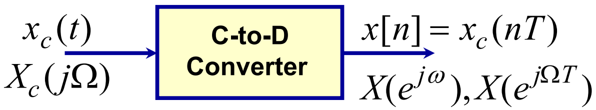
“Typical” bandlimited signal



Fourier transform of samples



Sampling Theorem



- Frequency-domain representation

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega T n} = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

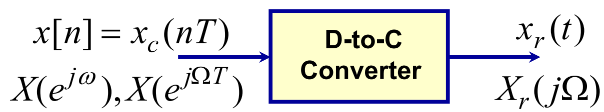
- Sampling theorem justification:

$$\text{If } X_c(j\Omega) = 0, \quad |\Omega| \geq \Omega_N \quad \text{and} \quad \Omega_s/2 = \pi/T \geq \Omega_N,$$

$$\text{then } X(e^{j\Omega T}) = \frac{1}{T} X_c(j\Omega), \quad |\Omega| \leq \pi/T = \Omega_s/2$$

Therefore we should be able to recover $x_c(t)$!

Bandlimited Reconstruction



$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT)$$

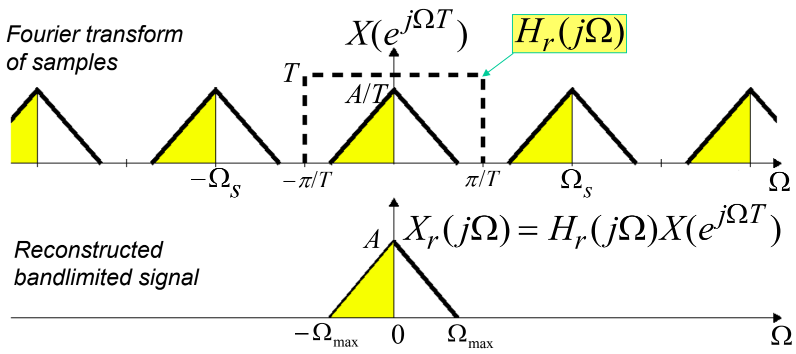
- Frequency-domain representation:

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] \left(e^{-j\Omega T n} H_r(j\Omega) \right)$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega T n} \right) H_r(j\Omega) = X(e^{j\Omega T}) H_r(j\Omega)$$

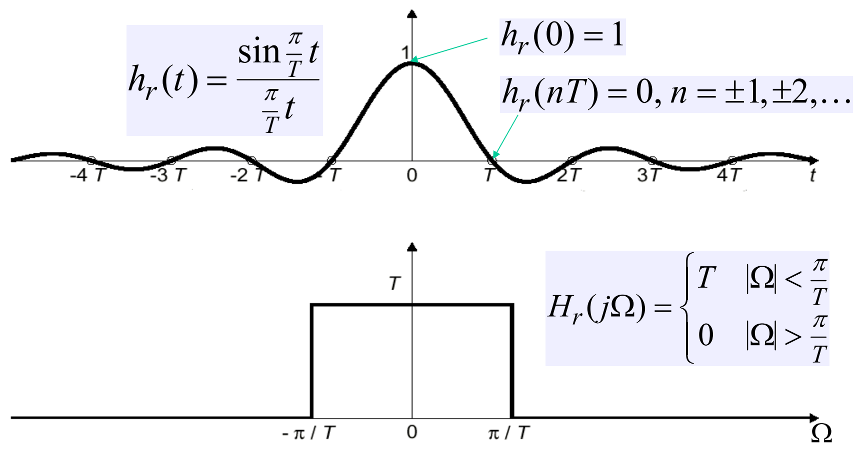
$$= H_r(j\Omega) \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k/T))$$

Reconstruction (Frequency-Domain)



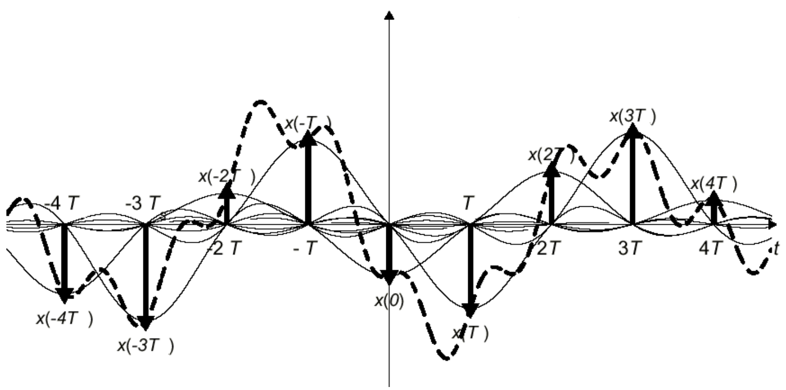
- If $\Omega_s > 2\Omega_{\max}$, the copies of $X_c(j\Omega)$ do not overlap, so $X_r(j\Omega) = H_r(j\Omega)\, X(e^{j\Omega T}) = X_c(j\Omega)$, and we get perfect reconstruction.

Reconstruction Filter

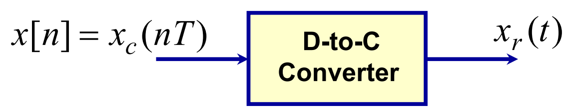


Reconstruction (Time-Domain)

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)h_r(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$



Signal Reconstruction



$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT)$$

- Types of interpolation pulses:
  - Square pulse -- holds sample value
  - Triangular pulse -- linear interpolation
  - Sinc pulse -- ideal bandlimited interpolation

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

D-to-A Conversion

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n]h_0(t-nT), \text{ where } h_0(t) = \begin{cases} 1, & -T/2 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases}$$

$$X_{DA}(j\Omega) = \hat{X}(e^{j\Omega T})\underbrace{\frac{2\sin(\Omega T/2)}{\Omega}}_{H_0(j\Omega)}$$

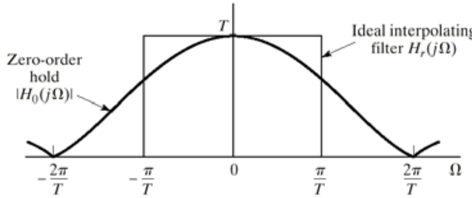
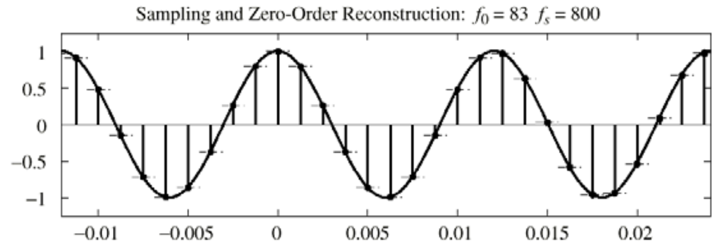
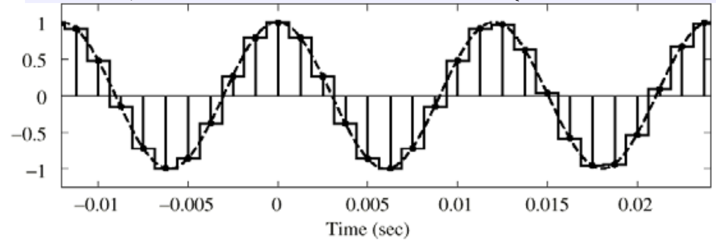


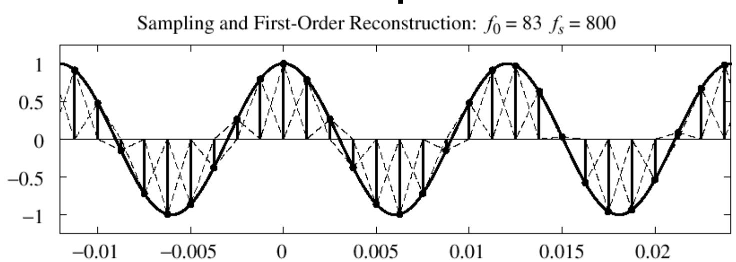
Illustration of D-to-A Conversion



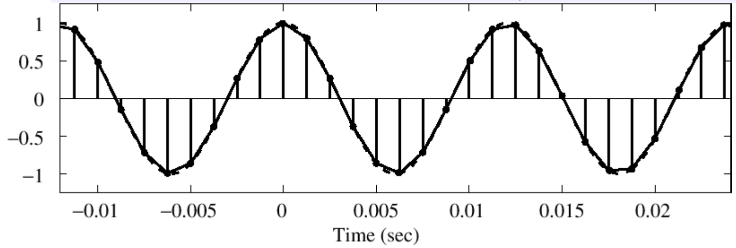
$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n]h_0(t-nT), \text{ where } h_0(t) = \begin{cases} 1, & -T/2 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases}$$



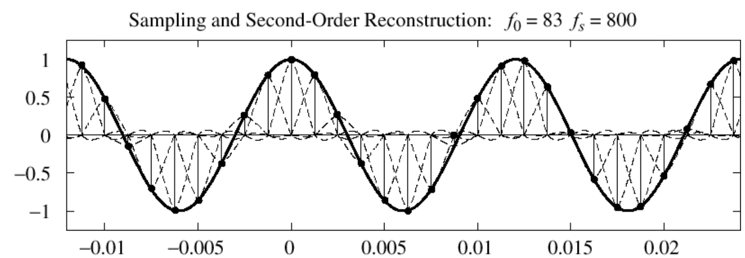
Linear Interpolation



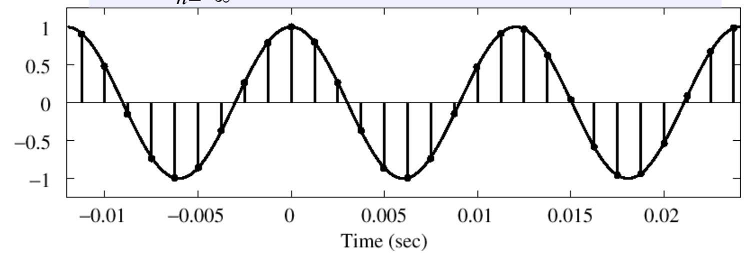
$$x_r(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n]h_l(t-nT), \text{ where } h_l(t) = \begin{cases} 1-|t/T|, & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$



Ideal Interpolation

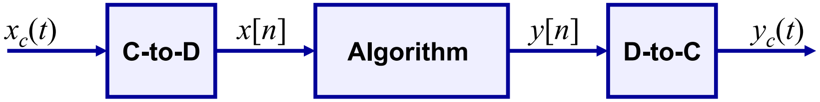


$$x_r(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n]h_r(t-nT), \text{ where } h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$



Idealized System (DSP Theory)

- **A-to-D conversion --> C-to-D conversion**
- **Finite precision arithmetic --> real numbers**
- **D-to-A conversion --> D-to-C conversion**



Summary

A bandlimited signal with highest frequency component at $\Omega_{\max}$ can be reconstructed exactly from samples taken with sampling frequency Ω_s (radians/s)

$$\Omega_s = \frac{2\pi}{T} > 2\Omega_{\max}$$

- If an analog signal is not bandlimited, it must be lowpass-filtered before sampling in order to avoid distortion by aliasing.
- Filtering specifications can be relaxed by oversampling.