ECE 4270

Fundamentals of Digital Signal Processing

Lecture 9: Discrete-Time Processing of Continuous-Time Signals

School of Electrical and Computer Engineering
Georgia Institute of Technology
Summer, 2004

Review of the Relation between Fourier Transforms of Continuous and Discrete Signals

The Sampled Continuous-time Signal Provides the Link - 1

$$x_c(t) \Leftrightarrow x_s(t) \Leftrightarrow x[n]$$

Sampled continuous-time signal

- $X_s(j\Omega)$ in terms of $X_c(j\Omega)$:

$$x_s(t) = x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$X_S(j\Omega) = \frac{1}{2\pi}X_C(j\Omega) * \sum_{k=-\infty}^{\infty}\frac{2\pi}{T}\delta(\Omega - k\frac{2\pi}{T})$$

$$X_S(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_C(j\Omega - jk\frac{2\pi}{T})$$

The Sampled Continuous-time Signal Provides the Link - 2

- But $X_s(j\Omega)$ also looks like the DTFT of $x[n]$ with $\omega = \Omega T$:

$$x_s(t) = x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty}x_c(nT)\delta(t-nT)$$

$$X_S(j\Omega) = \int_{-\infty}^{\infty}x_s(t)e^{-j\Omega t}dt = \sum_{n=-\infty}^{\infty}x_c(nT)e^{-j\Omega nT}$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty}x[n]e^{-j(\Omega T)n}$$

DTFT of $x[n]$ with $\omega = \Omega T$

$$= X(e^{j\Omega T}) = X(j\omega)|_{\omega=\Omega T}$$

The Sampled Continuous-time Signal Provides the Link - 3

$$x_c(t) \Leftrightarrow x_s(t) \Leftrightarrow x[n]$$

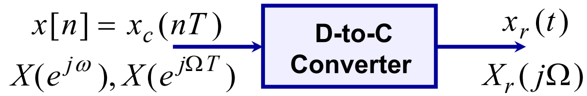
$$X(e^{j\Omega T}) = X_S(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_C(j\Omega - jk\frac{2\pi}{T})$$

or

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_C\left(\frac{1}{T}(\omega - 2\pi k)\right)$$

$$\Omega_s = 2\pi/T \qquad \omega = \Omega T$$

Bandlimited Reconstruction



$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$

- Frequency-domain representation

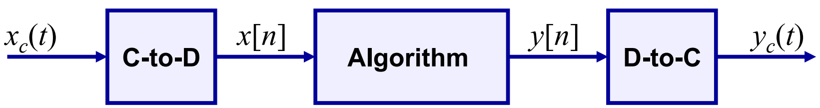
$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]\left(e^{-j\Omega Tn}H_r(j\Omega)\right)$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega Tn}\right)H_r(j\Omega) = X(e^{j\Omega T})H_r(j\Omega)$$

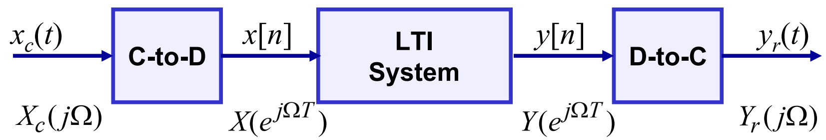
$$= \left(\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k/T))\right)H_r(j\Omega)$$

Idealized System (DSP Theory)

- A-to-D conversion --> C-to-D conversion
- Finite precision arithmetic --> real numbers
- D-to-A conversion --> D-to-C conversion



DT Filtering of CT Signals



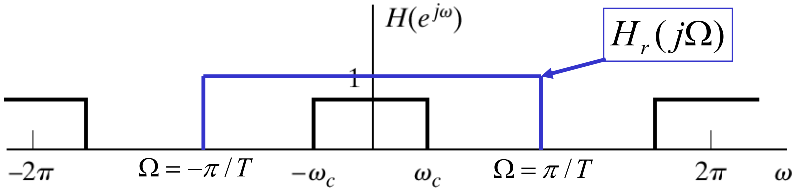
$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega Tn} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$Y(e^{j\Omega T}) = H(e^{j\Omega T})X(e^{j\Omega T})$$

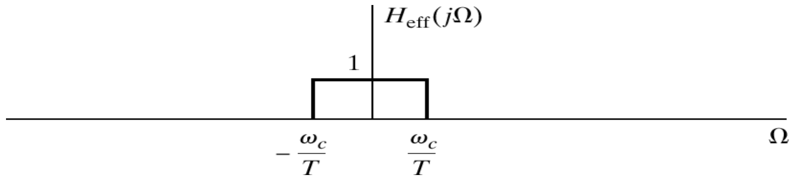
$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})$$

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

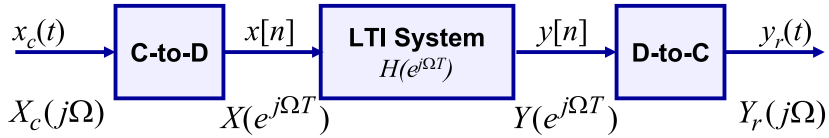
Effective Frequency Response



$$H_{\text{eff}}(j\Omega) = H(e^{j\Omega T}) \quad |\Omega| < \frac{\pi}{T}$$



DT Filtering of CT Signals



$$Y_r(j\Omega) = H(e^{j\Omega T})\left(\overset{1}{\cancel{H_r(j\Omega)}}\frac{1}{T}\right)\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

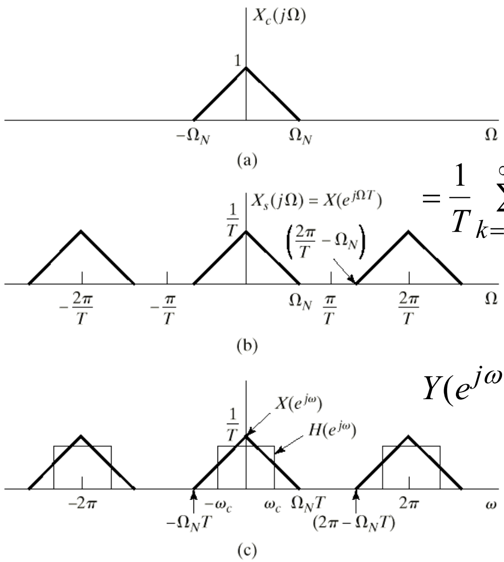
- If the input is bandlimited such that

$$X_c(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

and $2\pi/T \geq 2\Omega_N$, then the overall input and output are related by

$$Y_r(j\Omega) = H(e^{j\Omega T})X_c(j\Omega)$$

D-T Linear Filtering of C-T Signals - 1

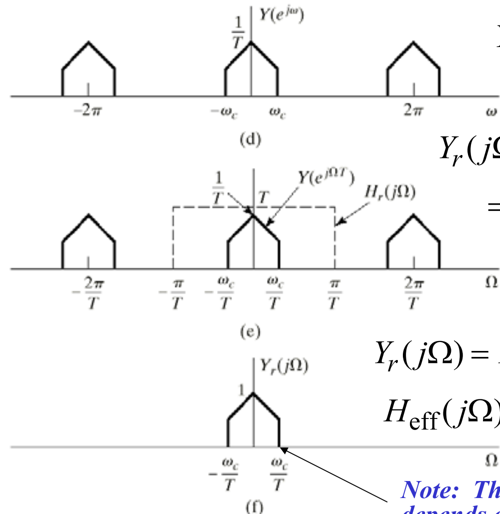


$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega + \frac{2\pi k}{T}\right)\right)$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$\omega = \Omega T$$

D-T Linear Filtering of C-T Signals - 2



$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})$$

$$= H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T})$$

$$\Omega = \frac{\omega}{T}$$

$$Y_r(j\Omega) = H(e^{j\Omega T})H_r(j\Omega)\frac{1}{T}X_c(j\Omega)$$

$$H_{\text{eff}}(j\Omega) = H(e^{j\Omega T}) \quad |\Omega| < \pi/T$$

Note: The effective cutoff frequency depends on the sampling rate!

Another Example

- Difference equation:

$$y[n] = ay[n-1] + bx[n]$$

- Frequency response:

$$H(e^{j\omega}) = \frac{b}{1 - ae^{-j\omega}}$$

- Overall frequency response

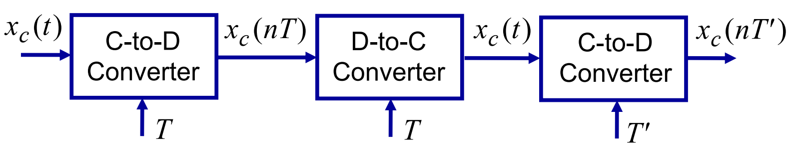
$$H(j\Omega) = H(e^{j\Omega T}) = \frac{b}{1 - ae^{-j\Omega T}} \qquad |\Omega| < \frac{\pi}{T}$$

Changing the Sampling Rate Digitally

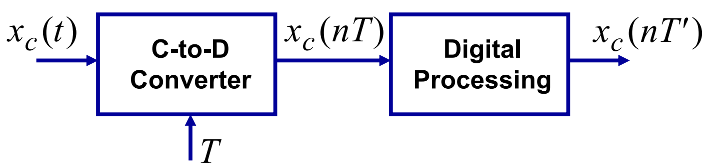
- The need to change sampling rates
- Decimation
- Interpolation
- Changing sampling rate by non-integer factors
- Oversampling eases anti-aliasing filtering

Sampling Rate Conversion

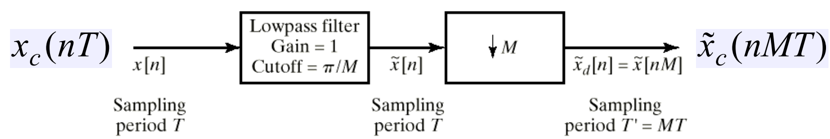
- In principle, so long as we have sampled at the Nyquist rate or above, we could do this:



- In practice, we can accomplish the same thing without converting back to analog:



Decimation - I



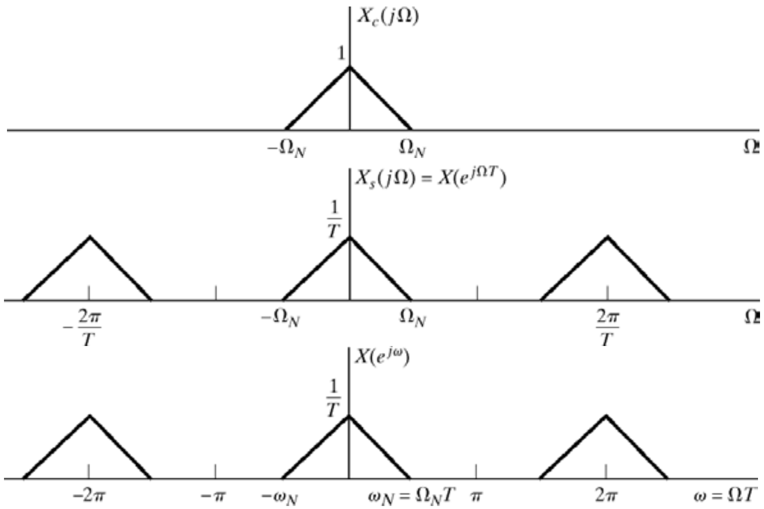
$$x[n] \Leftrightarrow X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$\tilde{x}[n] \quad \Leftrightarrow \quad \tilde{X}(e^{j\Omega T}) = H(e^{j\Omega T})X(e^{j\Omega T})$$

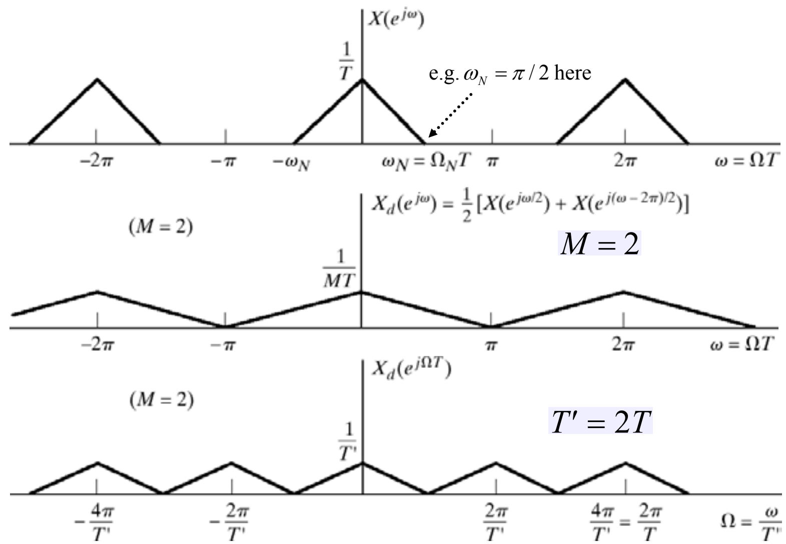
$$\tilde{x}_d[n] \Leftrightarrow \tilde{X}_d(e^{j\Omega MT}) = \frac{1}{MT}\sum_{k=-\infty}^{\infty} \tilde{X}_c\left(j(\Omega - \frac{2\pi k}{MT})\right)$$

$$= \frac{1}{M}\sum_{r=0}^{M-1} \tilde{X}(e^{j(\Omega T - 2\pi r)/M}) = \frac{1}{M}\sum_{r=0}^{M-1} \tilde{X}(e^{j(\omega - 2\pi r)/M})$$

Decimation - II

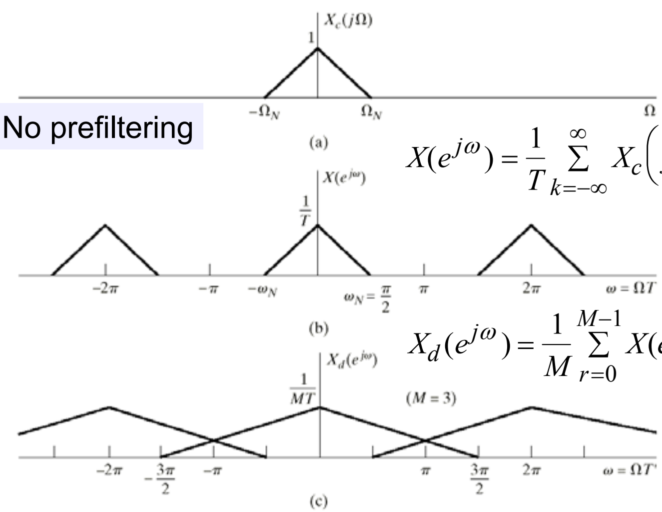


Decimation - III



Decimation - III

No prefiltering



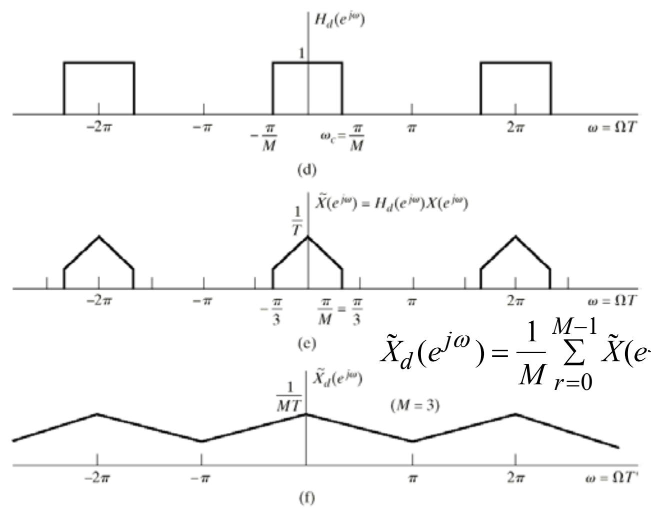
$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{r=0}^{M-1} X(e^{j(\omega-2\pi r)/M})$$

$M = 3$

$T' = 3T$

Decimation - IV

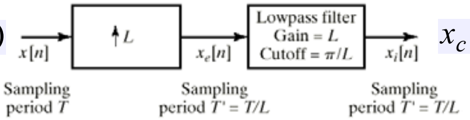


$M = 3$

$$\tilde{X}_d(e^{j\omega}) = \frac{1}{M}\sum_{r=0}^{M-1} \tilde{X}(e^{j(\omega-2\pi r)/M})$$

$T' = 3T$

Interpolation - I

$x_c(nT)$  $x_c(nT/L)$

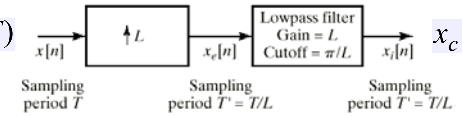
$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL] = \begin{cases} x[n/L], & n = 0, \pm L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n-kL)/L]}{[\pi(n-kL)/L]} \quad \text{or since}$$

$$x_c(t) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(t-kT)/T]}{[\pi(t-kT)/T]}$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(nT/L-kT)/T]}{[\pi(nT/L-kT)/T]} = x_c(nT/L)$$

Interpolation - II

$x_c(nT)$  $x_c(nT/L)$

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]e^{-j\omega n}$$

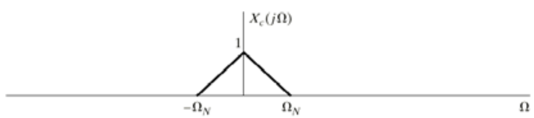
$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega Lk} = X(e^{j\omega L})$$

$$X_e(e^{j\Omega T/L}) = X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$X_i(e^{j\Omega T/L}) = H_i(e^{j\Omega T/L})X_e(e^{j\Omega T/L})$$

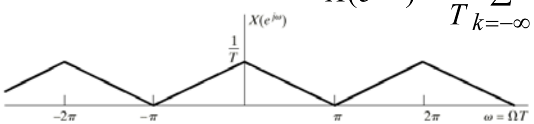
$$= \frac{1}{(T/L)}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T/L}\right)\right)$$

Interpolation - III

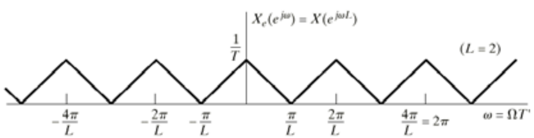


(a)

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$



(b)

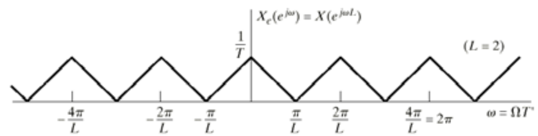


(c)

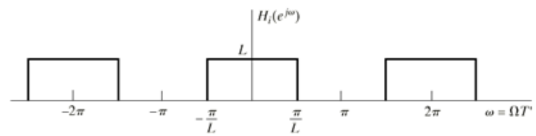
$L = 2$

$T' = T/2$

Interpolation - IV

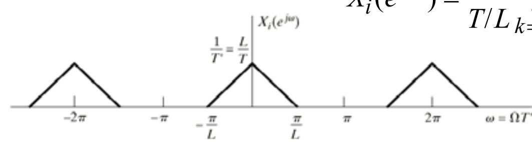


(c)



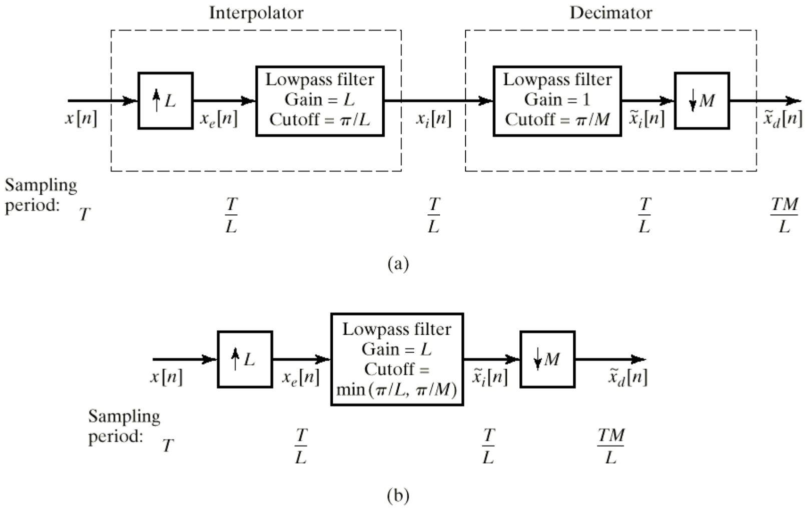
(d)

$$X_i(e^{j\omega}) = \frac{1}{T/L}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T/L} - \frac{2\pi k}{T/L}\right)\right)$$

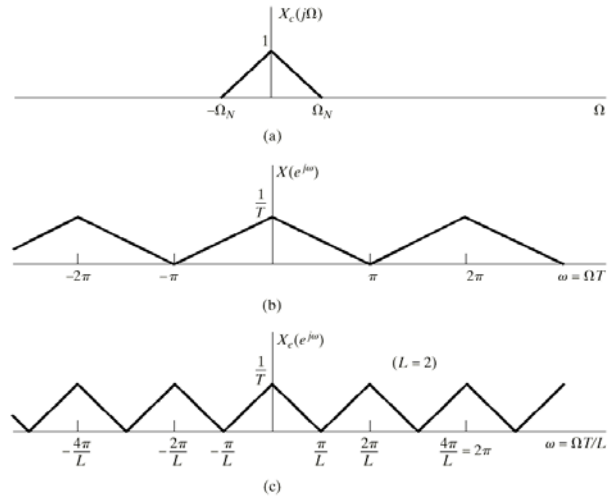


(e)

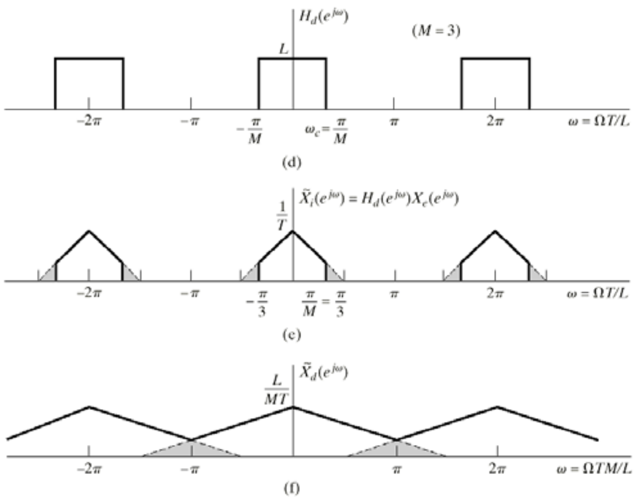
Non-Integer Rate Change - I



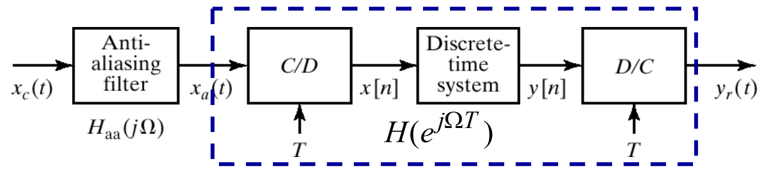
Non-Integer Rate Change - II



Non-Integer Rate Change - III



Anti-Alias Pre-filtering



$$Y_r(j\Omega) = H(e^{j\Omega T})X_a(j\Omega) \text{ if } X_a(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

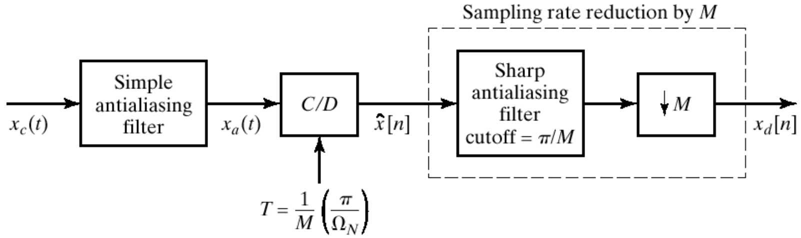
- What is the overall effective frequency response?

$$X_a(j\Omega) = H_{aa}(j\Omega)X_c(j\Omega)$$

$$\Rightarrow \quad X_a(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

$$Y_r(j\Omega) = \underbrace{H(e^{j\Omega T})H_{aa}(j\Omega)}_{H_{eff}(j\Omega)}X_c(j\Omega)$$

Oversampling Eases Filtering - I

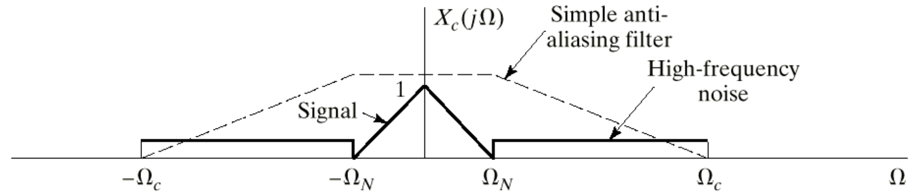


$$X_a(j\Omega) = H_{aa}(j\Omega)X_c(j\Omega)$$

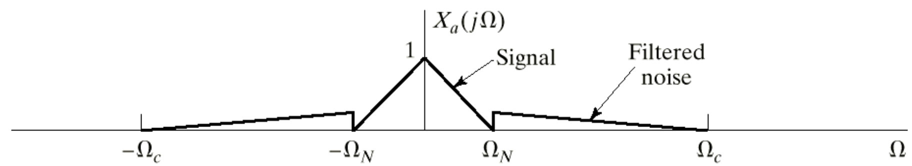
$$\text{Choose } H_{aa}(j\Omega) = 0 \ \text{ for } |\Omega| \geq M\Omega_N$$

$$\Rightarrow X_a(j\Omega) = 0 \ \text{ for } |\Omega| \geq M\Omega_N$$

Oversampling Eases Filtering - II

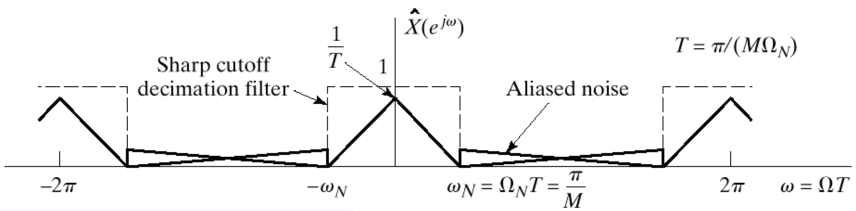


(a)



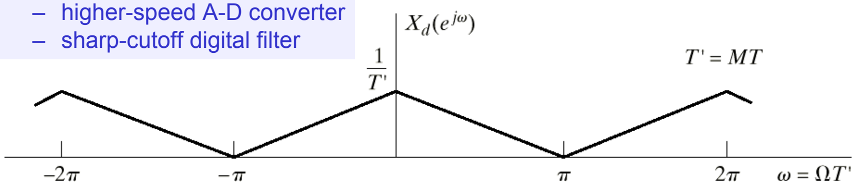
(b)

Oversampling Eases Filtering - III



(c)

- Complexity has been moved from analog anti-aliasing filter to
  - higher-speed A-D converter
  - sharp-cutoff digital filter



(d)