7.21. (a) Using the bilinear transform frequency mapping equation,

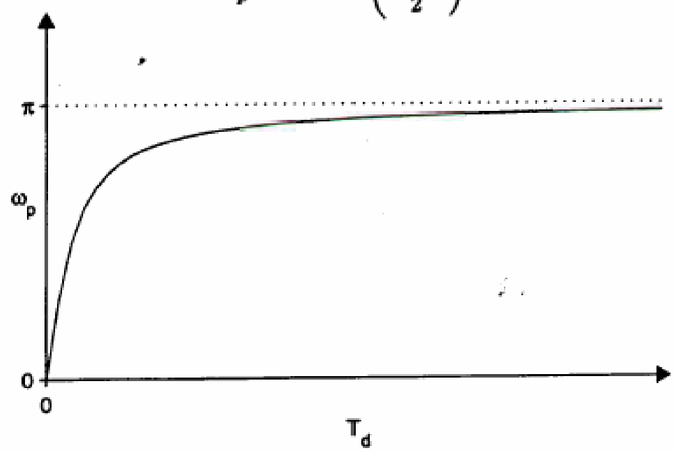
$$\Omega_p = \frac{2}{T_d}\tan\left(\frac{\omega_p}{2}\right)$$

we have

$$\begin{aligned} T_d &= \frac{2}{\Omega_p}\tan\left(\frac{\pi}{4}\right) \\ &= \frac{2}{\Omega_p} \end{aligned}$$

(b)

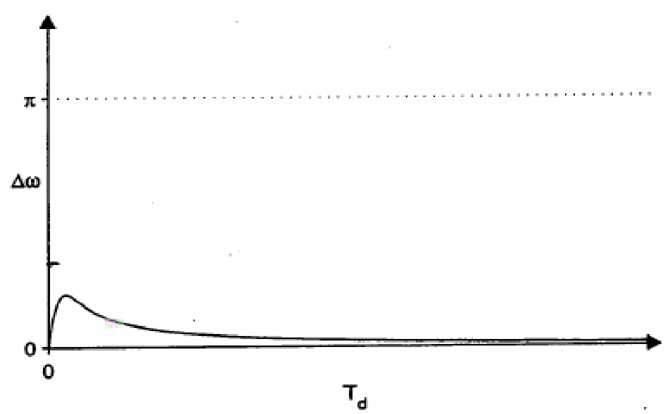
$$\omega_p = 2\tan^{-1}\left(\frac{\Omega_p T_d}{2}\right)$$



(c)

$$\begin{aligned} \omega_s &= 2\tan^{-1}\left(\frac{\Omega_s T_d}{2}\right) \\ \omega_p &= 2\tan^{-1}\left(\frac{\Omega_p T_d}{2}\right) \end{aligned}$$

$$\Delta\omega = \omega_s - \omega_p = 2\left[\tan^{-1}\left(\frac{\Omega_s T_d}{2}\right) - \tan^{-1}\left(\frac{\Omega_p T_d}{2}\right)\right]$$



7.22. (a) Applying the bilinear transform yields

$$\begin{aligned} H(z) &= H_c(s)\,\big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{T_d}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right), \qquad |z| > 1 \end{aligned}$$

which has the impulse response

$$h[n] = \frac{T_d}{2}\left(u[n] + u[n-1]\right)$$

(b) The difference equation is

$$y[n] = \frac{T_d}{2}\left(x[n] + x[n-1]\right) + y[n-1]$$

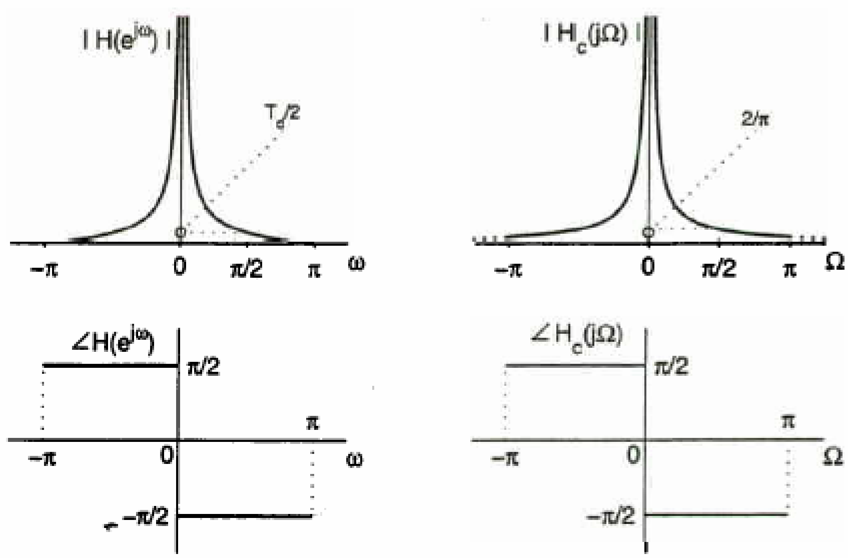
This system is not implementable since it has a pole on the unit circle and is therefore not stable.

(c) Since this system is not stable, it does not strictly have a frequency response. However, if we ignore this mathematical subtlety we get

$$\begin{aligned} H(e^{j\omega}) &= \frac{T_d}{2}\left(\frac{1+e^{-j\omega}}{1-e^{-j\omega}}\right) \\ &= \frac{T_d}{2}\left(\frac{e^{j\omega/2}+e^{-j\omega/2}}{e^{j\omega/2}-e^{-j\omega/2}}\right) \\ &= \frac{T_d}{2j}\cot(\omega/2) \end{aligned}$$

and since the Laplace transform evaluated along the $j\Omega$ axis is the continous-time Fourier transform we also have

$$H_c(j\Omega) = \frac{1}{j\Omega}$$



In general, we see that we will not be able to approximate the high frequencies, but we can approximate the lower frequencies if we choose $T_d = 4/\pi$.

(d) Applying the bilinear transform yields

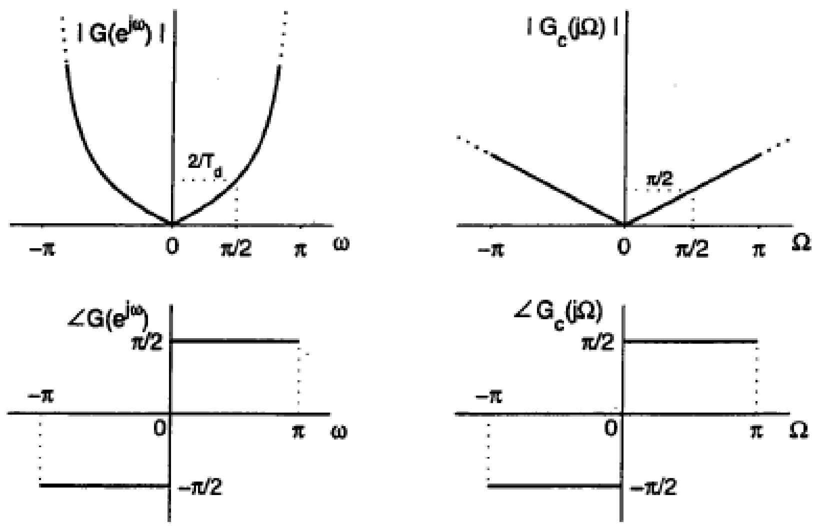
$$\begin{aligned} G(z) &= H_c(s)\,\Big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{2}{T_d}\left[\frac{1-z^{-1}}{1+z^{-1}}\right], \qquad |z| > 1 \end{aligned}$$

which has the impulse response

$$\begin{aligned} g[n] &= \frac{2}{T_d}\left[(-1)^n u[n] - (-1)^{n-1}u[n-1]\right] \\ &= \frac{2}{T_d}\left[2(-1)^n u[n] - \delta[n]\right] \end{aligned}$$

(e) This system does not strictly have a frequency response either, due to the pole on the unit circle. However, ignoring this fact again we get

$$\begin{aligned} G(e^{j\omega}) &= \frac{2}{T_d}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right] \\ &= \frac{2}{T_d}\left(\frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}}\right) \\ &= \frac{2j}{T_d}\tan(\omega/2) \\ G(j\Omega) &= j\Omega \end{aligned}$$

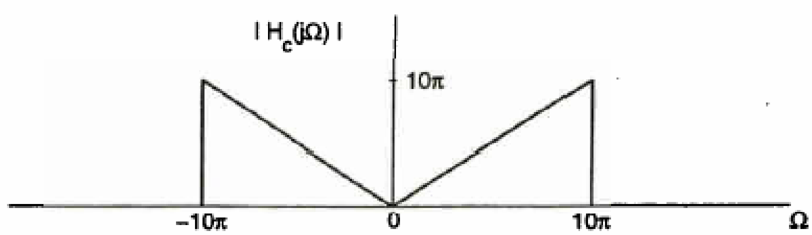


Again, we see that we will not be able to approximate the high frequencies, but we can approximate the lower frequencies if we choose $T_d = 4/\pi$.

(f) If the same value of T_d is used for each bilinear transform, then the two systems are inverses of each other, since then

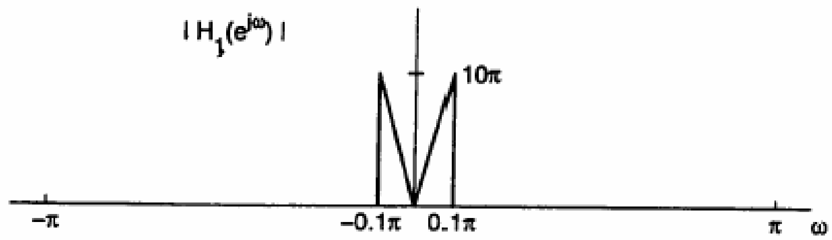
$$H(e^{j\omega})G(e^{j\omega}) = 1$$

7.23. We start with $|H_c(j\Omega)|$,



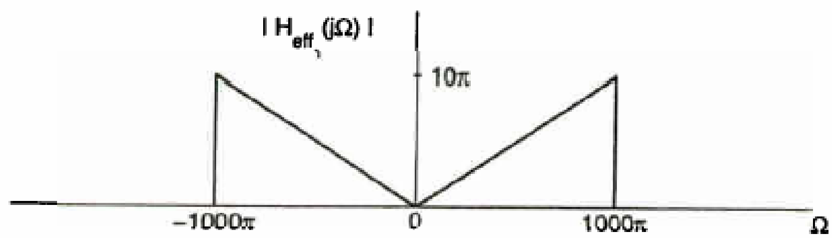
(a) By impulse invariance we scale the frequency axis by T_d to get

$$|H_1(e^{j\omega})| = \left|\sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi k}{T_d}\right)\right|$$



Then, to get the overall system response we scale the frequency axis by T and bandlimit the result according to the equation

$$|H_{\text{eff}_1}(j\Omega)| = \begin{cases} |H_1(e^{j\omega T})|, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$

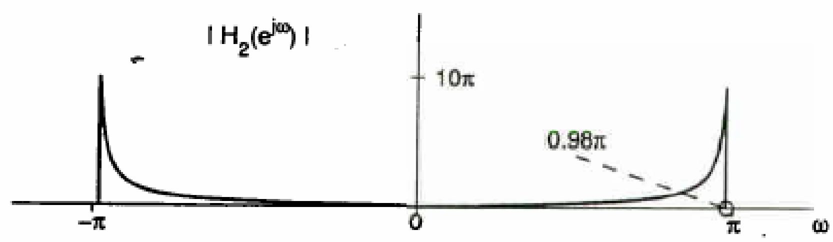


(b) Using the frequency mapping relationships of the bilinear transform,

$$\begin{aligned} \Omega &= \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right), \\ \omega &= 2\tan^{-1}\left(\frac{\Omega T_d}{2}\right), \end{aligned}$$

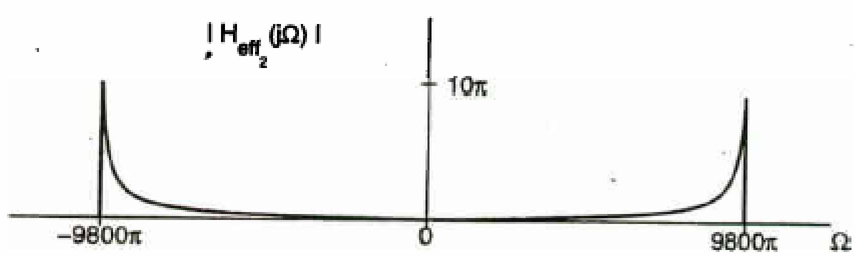
we get

$$|H_2(e^{j\omega})| = \begin{cases} |\tan\left(\frac{\omega}{2}\right)|, & |\omega| < 2\tan^{-1}(10\pi) = 0.98\pi \\ 0, & \text{otherwise} \end{cases}$$



Then, to get the overall system response we scale the frequency axis by T and bandlimit the result according to the equation

$$|H_{\text{eff}_2}(j\Omega)| = \begin{cases} |H_2(e^{j\omega T})|, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$



7.26. (a) Since

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)$$

and we desire

$$H(e^{j\omega})\,|_{\omega=0} = H_c(j\Omega)\,|_{\Omega=0},$$

we see that

$$H(e^{j\omega})|_{\omega=0} = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

requires

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = 0.$$

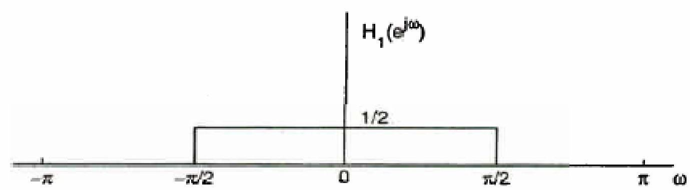
(b) Since the bilinear transform maps $\Omega = 0$ to $\omega = 0$, the condition will hold for any choice of $H_c(j\Omega)$.

7.27.

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| \leq \pi \end{cases}$$

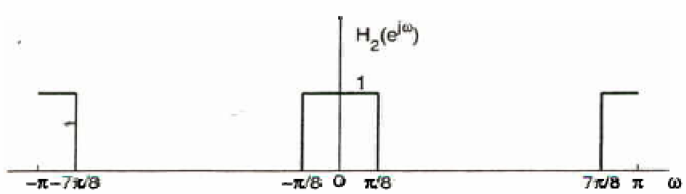
(a)

$$\begin{aligned} h_1[n] &= h[2n] \\ H_1(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[2n]e^{j\omega n} \\ &= \sum_{n \text{ even}} h[n]e^{\frac{j\omega n}{2}} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[h[n] + (-1)^n h[n]\right] e^{j\frac{\omega n}{2}} \\ &= \frac{1}{2}H(e^{j\frac{\omega}{2}}) + \frac{1}{2}H\left(e^{j\frac{\omega+2\pi}{2}}\right) \end{aligned}$$



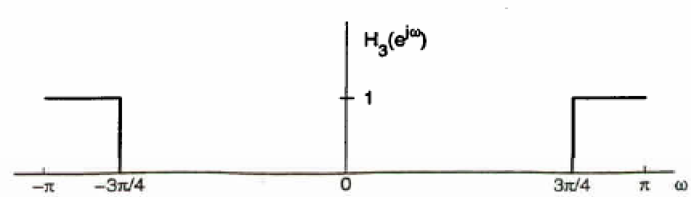
(b)

$$\begin{aligned} H_2(e^{j\omega}) &= \sum_{n \text{ even}} h[n/2]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega 2n} \\ &= H\left(e^{j2\omega}\right) \end{aligned}$$



(c)

$$H_3(e^{j\omega}) = H\left(e^{j(\omega+\pi)}\right)$$



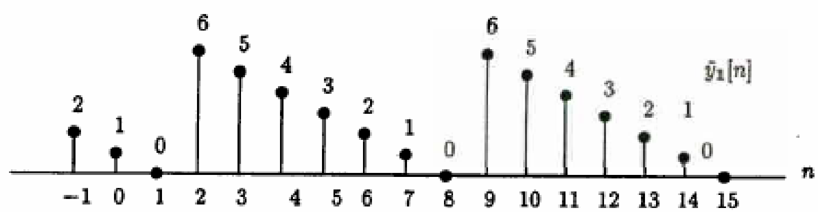
8.21. (a) We seek a sequence $\tilde{y}_1[n]$ such that

$$\tilde{Y}_1[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

From the discussion of Section 8.2.5, $\tilde{y}[n]$ is the result of the periodic convolution between $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$.

$$\tilde{y}_1[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

Since $\tilde{x}_2[n]$ is a periodic impulse, shifted by two, the resultant sequence will be a shifted (by two) replica of $\tilde{x}_1[n]$.



Using the analysis equation of Eq. (8.11), we may rigorously derive $\tilde{y}_1[n]$:

$$\begin{aligned}
\tilde{X}_1[k] &= \sum_{n=0}^{6} \tilde{x}_1[n]W_7^{kn} \\
&= 6 + 5W_7^{k} + 4W_7^{2k} + 3W_7^{3k} + 2W_7^{4k} + W_7^{5k} \\
\tilde{X}_2[k] &= \sum_{n=0}^{6} \tilde{x}_2[n]W_7^{kn} \\
&= W_7^{2k} \\
\tilde{Y}_1[k] &= \tilde{X}_1[k]\tilde{X}_2[k] \\
&= 6W_7^{2k} + 5W_7^{3k} + 4W_7^{4k} + 3W_7^{5k} + 2W_7^{6k} + W_7^{7k}
\end{aligned}$$

Noting that $W_7^{7k} = e^{j\frac{2\pi}{7}(7k)} = 1 = W_7^{0k}$, we use the synthesis equation of Eq. (8.12) to construct $\tilde{y}_1[n]$. The result is identical to the sequence depicted above.

(b) The DFS of the signal illustrated in Fig. P8.21-2.is given by:

$$\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{6} \tilde{x}_3[n]W_7^{kn} \\
&= 1 + W_7^{4k}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\tilde{Y}_2[k] &= \tilde{X}_1[k]\tilde{X}_3[k] \\
&= \tilde{X}_1[k] + W_7^{4k}\tilde{X}_1[k]
\end{aligned}$$

Since the DFS is linear, the inverse DFS of $\tilde{Y}_2[k]$ is given by:

$$\tilde{y}_2[n] = \tilde{x}_1[n] + \tilde{x}_1[n-4].$$

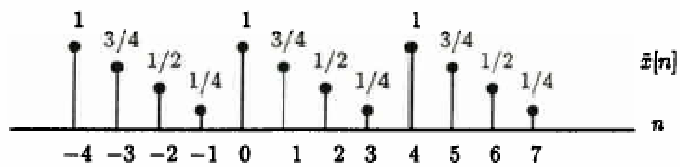
8.25. Both sequences $x[n]$ and $y[n]$ are of finite-length ($N = 4$).

Hence, no aliasing takes place. From Section 8.6.2, multiplication of the DFT of a sequence by a complex exponential corresponds to a circular shift of the time-domain sequence.

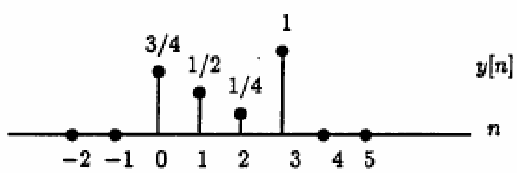
Given $Y[k] = W_4^{3k} X[k]$, we have

$$y[n] = x[((n-3))_4]$$

We use the technique suggested in problem 8.28. That is, we temporarily extend the sequence such that a periodic sequence with period 4 is formed.



Now, we shift by three (to the right), and set all values outside $0 \le n \le 3$ to zero.



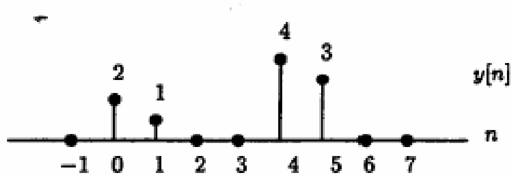
8.26. (a) When multiplying the DFT of a sequence by a complex exponential, the time-domain signal undergoes a circular shift.

For this case,

$$Y[k] = W_6^{4k} X[k], \quad 0 \le k \le 5$$

Therefore,

$$y[n] = x[((n-4))_6], \quad 0 \le n \le 5$$



(b) There are two ways to approach this problem. First, we attempt a solution by brute force.

$$\begin{aligned} X[k] &= 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}, \quad W_6^k = e^{-j(2\pi k/6)} \text{ and } 0 \le k \le 5 \\ W[k] &= \mathcal{R}e\{X[k]\} \\ &= \frac{1}{2}\left(X[k] + X^*[k]\right) \\ &= \frac{1}{2}\left(4 + 3W_6^k + 2W_6^{2k} + W_6^{3k} + 4 + 3W_6^{-k} + 2W_6^{-2k} + W_6^{-3k}\right) \end{aligned}$$

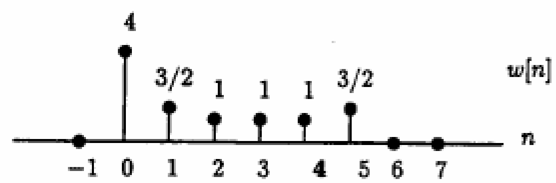
Notice that

$$
\begin{aligned}
W_N^k &= e^{-j(2\pi k/N)} \\
W_N^{-k} &= e^{j(2\pi k/N)} = e^{-j(2\pi/N)(N-k)} = W_N^{N-k} \\
W[k] &= 4 + \frac{3}{2}\left[W_6^k + W_6^{6-k}\right] + \left[W_6^{2k} + W_6^{6-2k}\right] + \frac{1}{2}\left[W_6^{3k} + W_6^{6-3k}\right], \quad 0 \le k \le 5
\end{aligned}
$$

So,

$$
\begin{aligned}
w[n] &= 4\delta[n] + \frac{3}{2}\Big(\delta[n-1] + \delta[n-5]\Big) + \delta[n-2] + \delta[n-4] \\
&\quad + \frac{1}{2}\Big(\delta[n-3] + \delta[n-3]\Big) \\
w[n] &= 4\delta[n] + \frac{3}{2}\delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \frac{3}{2}\delta[n-5], \quad 0 \le n \le 5
\end{aligned}
$$

Sketching $w[n]$:

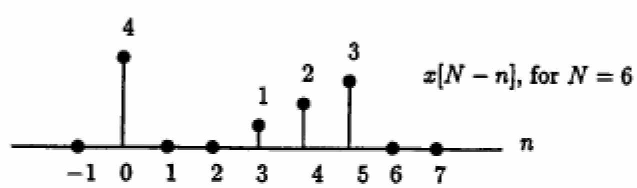


As an alternate approach, suppose we use the properties of the DFT as listed in Table 8.2.

$$
\begin{aligned}
W[k] &= \mathcal{R}e\{X[k]\} \\
&= \frac{X[k] + X^*[k]}{2} \\
w[n] &= \frac{1}{2}\,\text{IDFT}\{X[k]\} + \frac{1}{2}\,\text{IDFT}\{X^*[k]\} \\
&= \frac{1}{2}\Big(x[n] + x^*[((-n))_N]\Big)
\end{aligned}
$$

For $0 \le n \le N - 1$ and $x[n]$ real:

$$
w[n] = \frac{1}{2}\Big(x[n] + x[N-n]\Big)
$$



So, we observe that $w[n]$ results as above.

(c) The DFT is decimated by two. By taking alternate points of the DFT output, we have half as many points. The influence of this action in the time domain is, as expected, the appearance of aliasing. For the case of decimation by two, we shall find that an additional replica of x[n] surfaces, since the sequence is now periodic with period 3.

From part (b):

$$X[k] = 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}, \quad 0 \leq k \leq 5$$

Let $Q[k] = X[2k]$,

$$Q[k] = 4 + 3W_3^k + 2W_3^{2k} + W_3^{3k}, \quad 0 \leq k \leq 2$$

Noting that $W_3^{3k} = W_3^{0k}$

$$q[n] = 5\delta[n] + 3\delta[n-1] + 2\delta[n-2], \quad 0 \leq n \leq 2$$