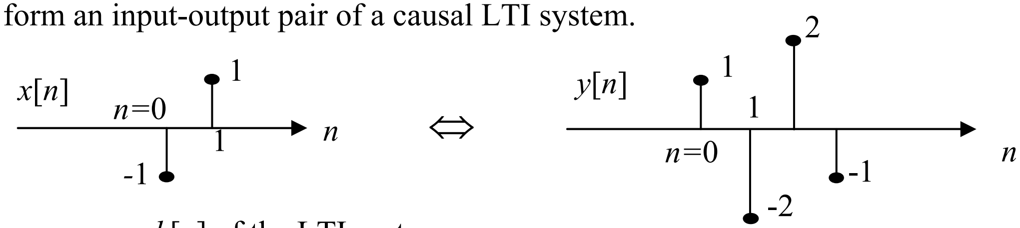
Problem 1 (30 points)

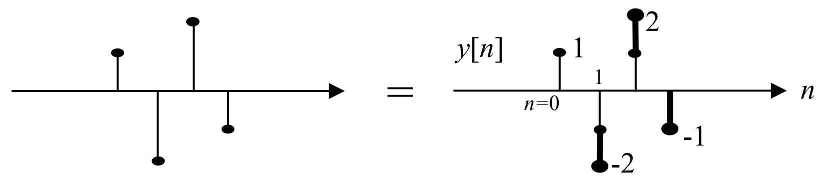
Signals $x[n]$ and $y[n]$ below form an input-output pair of a causal LTI system.



a) (10 pts) Find the impulse response $h[n]$ of the LTI system;
b) (10 pts) Find the frequency response of the system;
c) (10 pts) If the input to this system is $\sin(\omega n)$ where ω is a quarter of the (normalized) sampling rate, find the output, both in expression and in sketch.

Answer:

a)



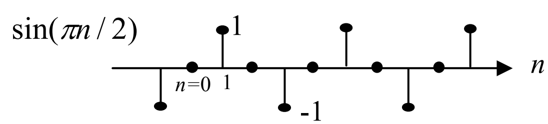
$$y[n] = \delta[n] - 2\delta[n-1] + 2\delta[n-2] - \delta[n-3],$$

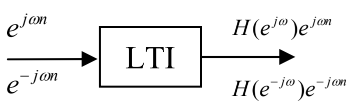
$$h[n] = -\delta[n] + \delta[n-1] - \delta[n-2]$$

This can be obtained either by inspection as sketched, or through convolution sum.

b) $H(e^{j\omega}) = -1 + e^{-j\omega} - e^{-j2\omega}$

c) Input is $\sin \omega n$ at $\omega = \frac{2\pi}{4} = \frac{\pi}{2}$





$$\sin \omega n = \frac{1}{2j}\left(e^{j\omega n} - e^{-j\omega n}\right),$$

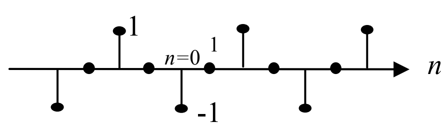
$$H(e^{j\omega}) = -1 + e^{-j\omega} - e^{-j2\omega}$$

$$H(e^{j\pi/2}) = -1 + e^{-j\pi/2} - e^{-j\pi} = -j$$

$$H(e^{-j\pi/2}) = -1 + e^{j\pi/2} - e^{j\pi} = j$$

Therefore,
The output is

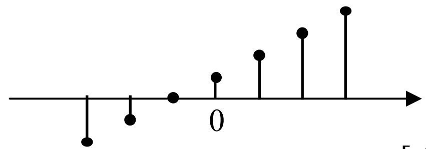
$$\frac{1}{2j}\left[H(e^{j\pi/2})e^{j\pi n/2} - H(e^{-j\pi/2})e^{-j\pi n/2}\right] = \frac{1}{2j}\left[-je^{j\pi n/2} - je^{-j\pi n/2}\right] = -\cos\left(\frac{\pi n}{2}\right)$$



This result should be identical to that obtained through convolution sum.

Problem 2 (10 points)

Find the Fourier Transform of the following discrete time signal. Use transform theorems and pairs (Table 2.2 and 2.3 in textbook) only and state how you use them. No direct expansion (you cannot afford the time anyway).



$x[-1]=0$, $x[0]= -x[-2]=0.5$, $x[1]= -x[-3]=1$, $x[2]=1.5$, $x[3]=2$, $x[n]=0$ for all other n

Answer:

Step 1 Use Table 2.3, #9 with $x_1[n]=\begin{cases}0.5, & 0\le n\le 6\\ 0, & \text{otherwise}\end{cases}$ and $X_1(e^{j\omega})=\dfrac{e^{-j3\omega}}{2}\dfrac{\sin\left(\frac{7\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$

Step 2 Use Table 2.3, #2 with to move the center back to the origin, i.e.,

$$x_2[n]=x_1[n+3] \quad \text{and} \quad X_2(e^{j\omega})=X_1(e^{j\omega})e^{j3\omega}=\frac{\sin\left(\frac{7\omega}{2}\right)}{2\sin\left(\frac{\omega}{2}\right)}$$

Step 3 Use Table 2.2, #5, $x_3[n]=nx_2[n]$, and $X_3(e^{j\omega})=j\dfrac{d}{d\omega}X_2(e^{j\omega})$

$$\frac{\sin\left(\frac{7\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}=1+2\cos\omega+2\cos 2\omega+2\cos 3\omega \qquad \frac{d}{d\omega}\frac{\sin\left(\frac{7\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}=-2\sin\omega-4\sin 2\omega-6\sin 3\omega$$

$$X_3(e^{j\omega})=-j(\sin\omega+2\sin 2\omega+3\sin 3\omega)$$

Step 4 $x[n]=x_2[n]+nx_2[n]=x_2[n]+x_3[n]$

Therefore, $X(e^{j\omega})=X_2(e^{j\omega})+X_3(e^{j\omega})=\dfrac{\sin\left(\frac{7\omega}{2}\right)}{2\sin\left(\frac{\omega}{2}\right)}-j(\sin\omega+2\sin 2\omega+3\sin 3\omega)$

Problem 3 (30 points)

The unit step sequence, $x[n]=u[n]$, is applied to an LTI system, producing the output $y[n]=\left(\frac{1}{2}\right)^{n-1}u[n+1]$.

a) (10 pts) Find $H(z)$, the z-transform of the system impulse response, and plot its pole-zero diagram;
b) (10 pts) Find the impulse response $h[n]$ of the system;
c) (5 pts) Is the system stable?
d) (5 pts) Is the system causal?

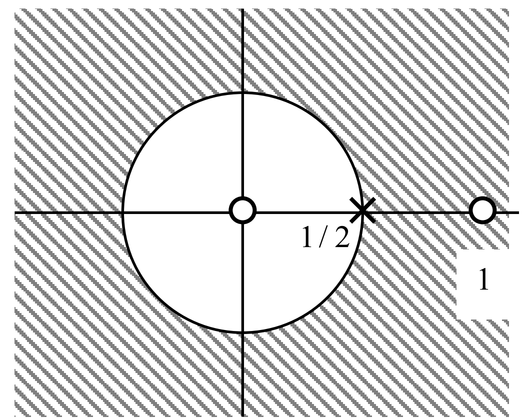
Answer:

$$x[n]=u[n] \Leftrightarrow X(z)=\frac{1}{1-z^{-1}} \qquad |z|>1$$

$$y[n]=\left(\frac{1}{2}\right)^{n-1}u[n+1]=4\left(\frac{1}{2}\right)^{n+1}u[n+1] \Leftrightarrow Y(z)=\frac{4z}{1-\frac{1}{2}z^{-1}} \qquad |z|>\frac{1}{2}$$

a)

$$H(z)=\frac{Y(z)}{X(z)}=\frac{4z(1-z^{-1})}{1-\frac{1}{2}z^{-1}} \qquad |z|>\frac{1}{2}$$



b)

$$H(z)=\frac{4z(1-z^{-1})}{1-\frac{1}{2}z^{-1}}=\frac{4z}{1-\frac{1}{2}z^{-1}}-\frac{4}{1-\frac{1}{2}z^{-1}} \qquad |z|>\frac{1}{2}$$

Therefore,

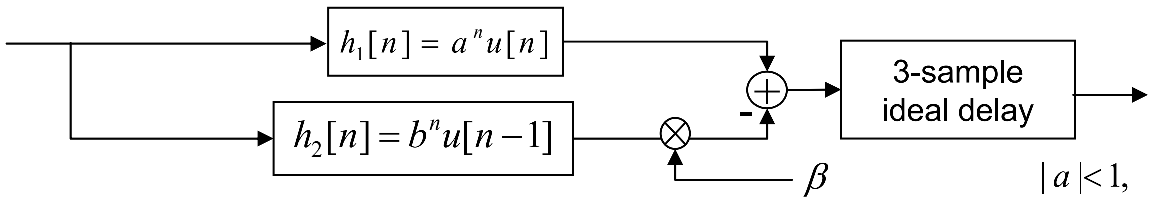
$$h[n]=4\left(\frac{1}{2}\right)^{n+1}u[n+1]-4\left(\frac{1}{2}\right)^{n}u[n]=4\delta[n+1]-2\left(\frac{1}{2}\right)^{n}u[n]$$

c) Yes, the ROC of H includes the unit circle and the system is stable.

d) No, because the impulse response is not zero for negative n.

Problem 4 (30 points)

The following block diagram defines a system. (see p.17 of text for ideal delay system)



$|a|<1, \quad |b|<1$

a) (10 pts) Find the overall impulse response $h[n]$ of the system (note the negative sign before summation)

b) (10 pts) Find the frequency response of the system

c) (10 pts) Find the condition (among a, b, and β) which will make the system an ideal delay.

Answer:

a) At the output of the adder, the impulse response is imply $h'[n] = h_1[n] - \beta h_2[n]$. After the 3-sample delay,

$$h[n] = h'[n-3] = h_1[n-3] - \beta h_2[n-3] = a^{n-3}u[n-3] - \beta b^{n-3}u[n-4]$$

b) $$H(e^{j\omega}) = \left[H_1(e^{j\omega}) - \beta H_2(e^{j\omega})\right]e^{-j3\omega}$$

$$H_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$ from Table 2.3, #4, and

$$H_2(e^{j\omega}) = \frac{be^{-j\omega}}{1-be^{-j\omega}}$$ from Table 2.2 and 2.3, and example 2.26 on page 63

Therefore, $$H(e^{j\omega}) = \left[\frac{1}{1-ae^{-j\omega}} - \frac{\beta be^{-j\omega}}{1-be^{-j\omega}}\right]e^{-j3\omega}$$

c) For to be a pure delay, the quantity in the bracket has to be unity or for some *m.*
When $a = b$ and $\beta = 1$, the system is indeed a pure delay of 3 samples because the bracketed term becomes unity.