ECE 4270 Final Exam
Summer 2004 - Prof. B.H. Juang
July 28, 2004

Student Name:_____________________

Test Duration: 2 hrs 50 mins

Problem 1

Consider a stable linear time-invariant system with impulse response $h_1[n]$. The input $x[n]$ and output $y[n]$ satisfy the difference equation: $$y[n-1]-\frac{17}{4}y[n]+y[n+1]=x[n]$$

a) Plot the poles and zeros of the system in the z-plane;

b) Find the impulse response $h_1[n]$.

Consider another stable linear time-invariant system with impulse response $h_2[n]$. The input $y[n]$ and output $v[n]$ satisfy the difference equation: $v[n]=y[n]-y[n-1]$. The overall system with impulse response $h[n]$ is formed by putting $h_1[n]$ and $h_2[n]$ in cascade.

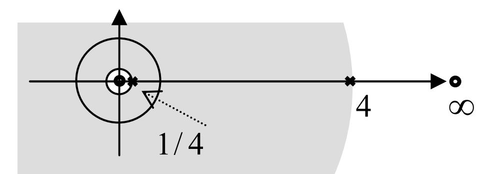
c) Plot the poles and zeros of the overall system in the z-plane;

d) Find the impulse response of the overall system, $h[n]$; (if your answer include overlapping step functions, reduce it for full credit)

e) If $x[n]=1+(-1)^n$, find $V(e^{j\omega})$, the discrete time Fourier transform of $v[n]$.

Answer: a) From the difference equation, $Y(z)(z^{-1}-\frac{17}{4}+z)=X(z)$. Therefore, $H_1(z)=\dfrac{1}{(z^{-1}-\frac{17}{4}+z)}=\dfrac{z}{(z^2-\frac{17}{4}z+1)}=\dfrac{z}{(z-\frac{1}{4})(z-4)}$

$H_1(z)$ has two poles at 4 and $\frac{1}{4}$, respectively and two zeros at 0 and ∞, respectively.



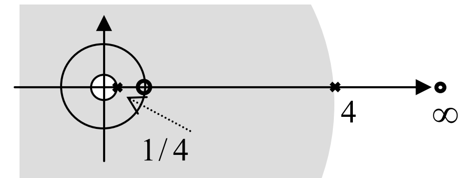
b) $$H_1(z)=\frac{z^{-1}}{(1-\frac{1}{4}z^{-1})(1-4z^{-1})}=\frac{A}{(1-\frac{1}{4}z^{-1})}+\frac{B}{(1-4z^{-1})} \quad \text{where } A=-\frac{4}{15} \text{ and } B=\frac{4}{15}$$

For $|z|>\frac{1}{4}$, $\dfrac{A}{(1-\frac{1}{4}z^{-1})} \Leftrightarrow A(\frac{1}{4})^n u[n]$ and for $|z|<4$, $\dfrac{B}{(1-4z^{-1})} \Leftrightarrow -B(4)^n u[-n-1]$

Therefore, $h[n]=-\frac{4}{15}(\frac{1}{4})^n u[n]-\frac{4}{15}(4)^n u[-n-1]$

c) $$Y(z)(1-z^{-1})=V(z) \qquad V(z)=(1-z^{-1})Y(z)=(1-z^{-1})\frac{X(z)}{(z^{-1}-\frac{17}{4}+z)} \qquad H(z)=\frac{V(z)}{X(z)}=\frac{z-1}{(z-\frac{1}{4})(z-4)}$$

$H(z)$ has two poles at 4 and $\frac{1}{4}$, respectively and two zeros at 1 and ∞, respectively.



d) $$H(z)=\frac{z^{-1}(1-z^{-1})}{(1-\frac{1}{4}z^{-1})(1-4z^{-1})}=-1+\frac{P}{(1-\frac{1}{4}z^{-1})}+\frac{Q}{(1-4z^{-1})} \quad \text{where } P=\frac{4}{5} \text{ and } Q=\frac{1}{5}$$

For $|z|>\frac{1}{4}$, $\dfrac{P}{(1-\frac{1}{4}z^{-1})} \Leftrightarrow P(\frac{1}{4})^n u[n]$ and for $|z|<4$, $\dfrac{Q}{(1-4z^{-1})} \Leftrightarrow -Q(4)^n u[-n-1]$

Therefore, $h[n]=-\delta[0]+\frac{4}{5}(\frac{1}{4})^n u[n]-\frac{1}{5}(4)^n u[-n-1]=\frac{4}{5}(\frac{1}{4})^n u[n-1]-\frac{1}{5}(4)^n u[-n]$

e) $x[n]=1+(-1)^n=1+\cos(n\pi)$ $\quad X(e^{j\omega})=\sum_{k=-\infty}^{\infty}2\pi\delta(\omega+2\pi k)+[\pi\delta(\omega-\pi+2\pi k)+\pi\delta(\omega+\pi+2\pi k)]$

$$H(e^{j\omega})=\frac{e^{j\omega}-1}{(e^{j\omega}-\frac{1}{4})(e^{j\omega}-4)}$$

$$V(e^{j\omega})=\sum_{k=-\infty}^{\infty}\frac{-8}{25}[\pi\delta(\omega-\pi+2\pi k)+\pi\delta(\omega+\pi+2\pi k)]$$

$H(e^{j0})=0;\ H(e^{j\pi})=-8/25=H(e^{-j\pi})$

Problem 2

A continuous time signal $x_c(t) = 2\sin(\Omega_0 t + \frac{\pi}{4})\cos(\Omega_0 t + \frac{\pi}{4})$, where Ω_0 is 1000 Hz or 2000π radian per second is being sampled at Ω_s=8000 Hz with sampling period T = 0.125 ms.

a) Write an expression for the sampled discrete-time sequence $x[n]$ corresponding to this signal and sketch such a sequence as a function of the integer index n.

b) Find the 8-point DFT of $x[n]$.

We'd like to change the sampling rate to 16000 Hz (i.e., sampling period T'= 0.0625 ms) through the up-sampling process depicted in Figure 4.24 with L = 2;

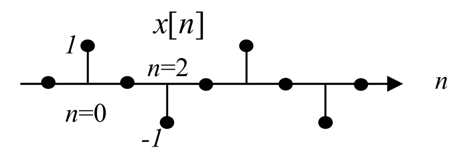
c) An up-sampled result $x_i[n]$ is obtained through linear interpolation (Eq. 4.92); Find the 8-point DFT of $x_i[n]$;

d) Find the 8-point DFT $D[k]$ of the interpolation error $d[n]$, $d[n] = x_c(nT') - x_i[n]$.

Answer:

a) $x_c(t) = 2\sin(\Omega_0 t + \frac{\pi}{4})\cos(\Omega_0 t + \frac{\pi}{4}) = \sin(2\Omega_0 t + \frac{\pi}{2}) = \cos(2\Omega_0 t)$

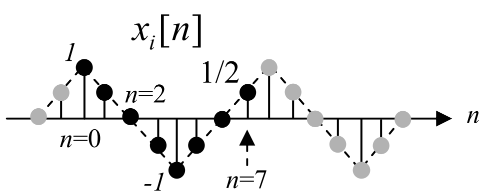
$2\Omega_0 T = 2\pi \bullet 2 \bullet 1000/8000 = \pi/2, \quad \therefore x[n] = x_c(nT) = \cos 2\Omega_0 nT = \cos\frac{n\pi}{2}$



b) $X[k] = \sum_{n=0}^{7} x[n] W_8^{kn} = W_8^0 - W_8^{2k} + W_8^{4k} - W_8^{6k} = W_4^0 - W_4^k + W_4^{2k} - W_4^{3k}$ Note: $W_4^0 = 1, W_4^1 = -j, W_4^2 = -1, W_4^3 = j$

$X[0] = 0, X[1] = 0, X[2] = 4, X[3] = 0, X[4] = 0, X[5] = 0, X[6] = 4, X[7] = 0$

c)



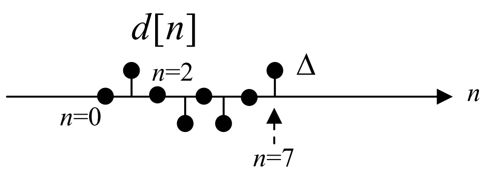
$X_i[k] = \sum_{n=0}^{7} x_i[n] W_8^{kn} = W_8^0 + \frac{1}{2}W_8^k - \frac{1}{2}W_8^{3k} - W_8^{4k} - \frac{1}{2}W_8^{5k} + \frac{1}{2}W_8^{7k}$

$= 1 - W_2^k + \frac{1}{2}W_8^k\left[1 - W_8^{2k} - W_8^{4k} + W_8^{6k}\right] = 1 - (-1)^k + \frac{1}{2}W_8^k\left[1 - W_4^k - W_4^{2k} + W_4^{3k}\right]$

$X_i[0] = 0, X_i[1] = 2 + \sqrt{2}, X_i[2] = 0, X_i[3] = 2 - \sqrt{2},$

$X_i[4] = 0, X_i[5] = 2 - \sqrt{2}, X_i[6] = 0, X_i[7] = 2 + \sqrt{2}$

d)



$\Delta = \frac{\sqrt{2}}{2} - \frac{1}{2} \approx 0.207$

For $n = 0, 2, 4, 6,\ x_c(nT') = x_i[n]$ and $d[n] = 0$.

For $n = 1$ and 7, $x_c(nT') = \frac{\sqrt{2}}{2}$ and $x_i[n] = \frac{1}{2}, \quad \therefore d[n] = \Delta$.

For $n = 3$ and 5, $x_c(nT') = -\frac{\sqrt{2}}{2}$ and $x_i[n] = -\frac{1}{2}, \quad \therefore d[n] = -\Delta$.

$D[k] = \sum_{n=0}^{7} d[n] W_8^{kn} = \Delta(W_8^k - W_8^{3k} - W_8^{5k} + W_8^{7k}) = \Delta W_8^k(1 - W_8^{2k} - W_8^{4k} + W_8^{6k}) = \Delta W_8^k(1 - W_4^k - W_4^{2k} + W_4^{3k})$

$D[0] = 0, D[1] = 2 - \sqrt{2}, D[2] = 0, D[3] = -2 + \sqrt{2}, D[4] = 0, D[5] = -2 + \sqrt{2}, D[6] = 0, D[7] = 2 - \sqrt{2}$

Or, $x_c(nT') = \cos(2\Omega_0 nT') = \cos\frac{n\pi}{4} = x''[n]$ which has the 8-point DFT: $X''[k] = 4$ for $k = 1,7$ and $X''[k] = 0$ for other k because of the freq. $\frac{\pi}{4}$.

Since $D[k] = X''[k] - X_i[k]$, we have

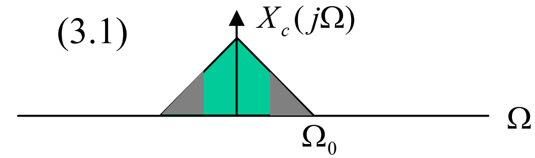
$D[0] = 0, D[1] = 4 - (2 + \sqrt{2}) = 2 - \sqrt{2}, D[2] = 0, D[3] = -X_i[3] = -2 + \sqrt{2}, D[4] = 0, D[5] = -X_i[5] = -2 + \sqrt{2}, D[6] = 0, D[7] = 4 - (2 + \sqrt{2}) = 2 - \sqrt{2}$

Problem 3

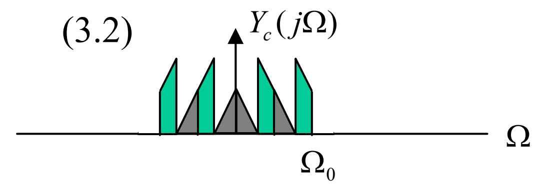
Figure 3.1 shows a typical spectrum (magnitude) of a continuous time signal $x_c(t)$. (Notice that shade is used to differentiate the high from the low frequency components.) Let $x[n]$ be a discrete-time version of $x_c(t)$, sampled at the rate of Ω_s .

a) Design a discrete-time processing scheme involving $x[n]$ that produces a continuous time output $y_c(t)$ that has a spectrum as shown in figure 3.2.

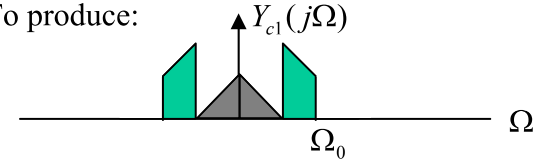
(3.1)



(3.2)



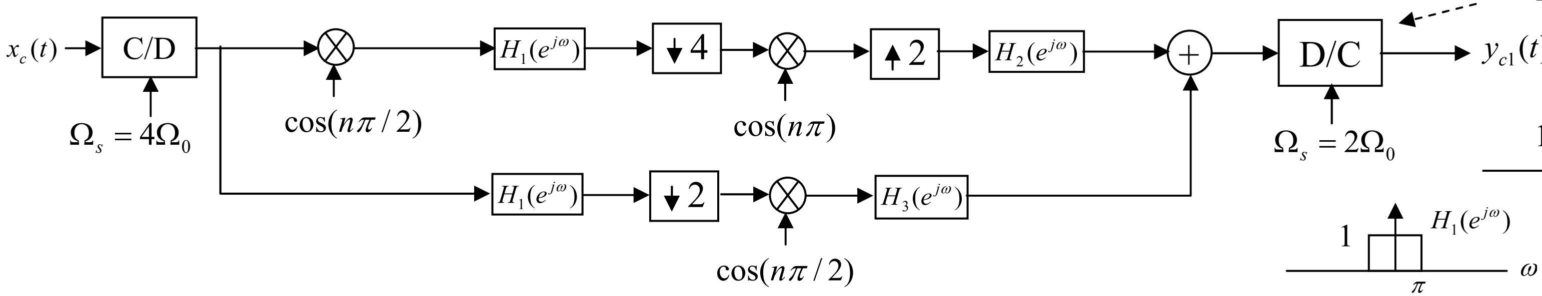
Answer: To produce:

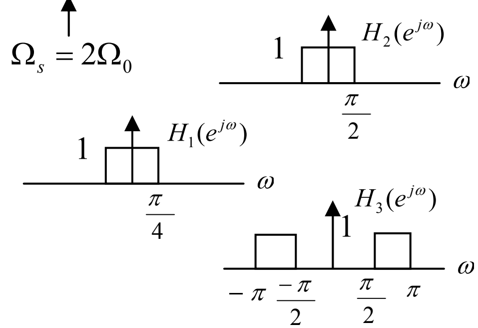


Use: (see Test #2)

Scheme A

including an ideal reconstruction filter with cutoff frequency at Ω_0





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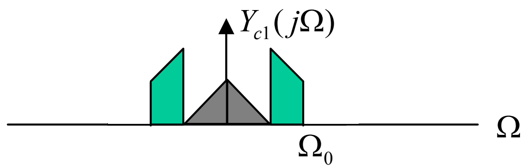

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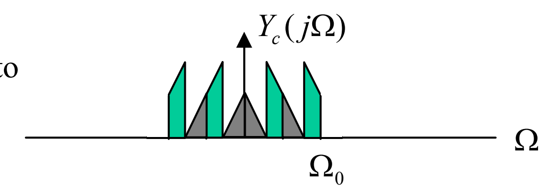
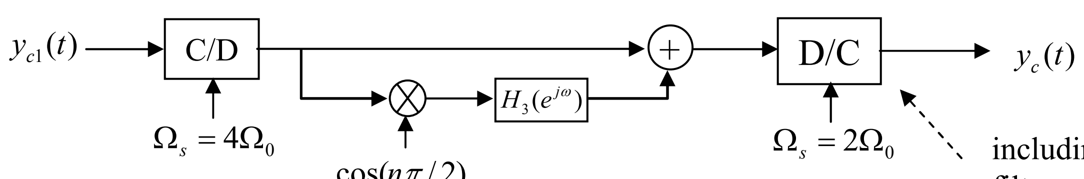


Use



including an ideal reconstruction filter with cutoff frequency at Ω_0

Problem 4

A signal $x[n]$ has the following discrete time Fourier transform (DTFT): $X(e^{j\omega}) = 1 + a\cos\omega + b\cos 2\omega$

a. Find the signal $x[n]$. Write your answer in impulse representation;

b. Let $X_s[k],\ k = 0, 1, 2, 3$ be the 4-point DFT of $x[n]$ obtained by sampling the DTFT. Find the corresponding signal $x_s[n]$; Write your answer for $n = 0, 1, 2, \ldots, 7$.

c. Let $X'[k] = X_s[((k))_4]$ for k=0, 1, 2, …, 7.. Find the signal $x'[n], n = 0, 1, 2, \ldots, 7$ that corresponds to $X'[k]$.

d. Find the most compact DFT $X[k],\ k = 0,1,2,.., K$ where K is the smallest number that allows undistorted reconstruction of $x[n]$;

e. The function $|X(e^{j\omega})|^2$ is called the energy density spectrum and the total energy of the sequence is defined as $E = (2\pi)^{-1}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega$; Find E and discuss its relationship with your answer for part (d).

Answer:

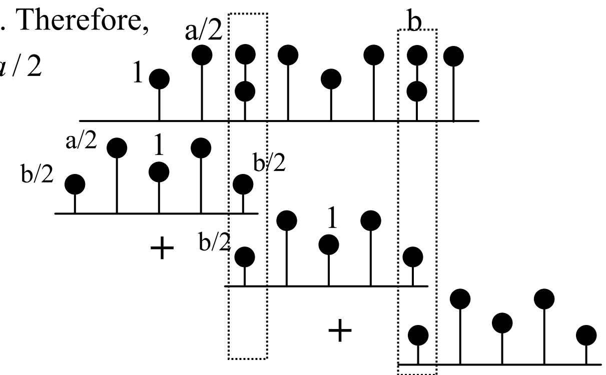
a) $$X(e^{j\omega}) = 1 + a\cos\omega + b\cos 2\omega = 1 + a\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + b\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right) = \frac{b}{2}e^{j2\omega} + \frac{a}{2}e^{j\omega} + 1 + \frac{a}{2}e^{-j\omega} + \frac{b}{2}e^{-j2\omega}$$

Therefore, $$x[n] = \frac{b}{2}\delta[n+2] + \frac{a}{2}\delta[n+1] + \delta[n] + \frac{a}{2}\delta[n-1] + \frac{b}{2}\delta[n-2]$$

b) $$X_s[k] = X(e^{j\omega})\big|_{\omega = 2\pi k/4} \quad \therefore X_s[0] = X(e^{j0}) = 1 + a + b,\ X_s[1] = X(e^{j\pi/2}) = 1 - b,\ X_s[2] = X(e^{j\pi}) = 1 - a + b,\ X_s[3] = X(e^{j3\pi/2}) = 1 - b$$

But we can consider $x[n]$ to be a finite sequence of length 5. Sampling its DTFT at 4 points around the unit circle corresponds to looking at a version of the original signal, repeated and aliased every 4 samples. Therefore,

$$x_s[0] = 1,\ x_s[1] = a/2,\ x_s[2] = b,\ x_s[3] = a/2,\ x_s[4] = 1,\ x_s[5] = a/2,\ x_s[6] = b,\ x_s[7] = a/2$$



c) $X'[k] = X_s[((k))_4] = X'[k+4]$

Therefore, $$x'[n] = \frac{1}{8}\sum_{k=0}^{7} X'[k]W_8^{nk} = \frac{1}{8}\sum_{k=0}^{3} X'[k]\left(W_8^{nk} + W_8^{n(k+4)}\right)$$

$$= \frac{1}{8}\sum_{k=0}^{3} X'[k]\left(W_8^{nk} + e^{-j\pi n}W_8^{nk}\right) = \frac{1}{8}\sum_{k=0}^{3} X'[k]W_8^{nk}\left(1 + e^{-j\pi n}\right)$$

When n is even, $x'[n] = x_s[n]$; when n is odd, $x'[n] = 0$.

d) Since $x[n]$ is a finite sequence of length 5, K=4. We can obtain its 5-DFT by sampling its DTFT at $2k\pi/5,\ k = 0,1,2,3,4$

$$X_5[k] = X(e^{j\omega})\big|_{\omega = 2k\pi/5} \quad X_5[0] = 1 + a + b,\ X_5[1] = 1 + a\cos\left(\frac{2\pi}{5}\right) + b\cos\left(\frac{4\pi}{5}\right),\ X_5[2] = 1 + a\cos\left(\frac{4\pi}{5}\right) + b\cos\left(\frac{8\pi}{5}\right),$$

$$X_5[3] = 1 + a\cos\left(\frac{6\pi}{5}\right) + b\cos\left(\frac{2\pi}{5}\right),\ X_5[4] = 1 + a\cos\left(\frac{8\pi}{5}\right) + b\cos\left(\frac{6\pi}{5}\right)$$

e) From the Parseval Theorem and (a),

$$E = (2\pi)^{-1}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega = \sum_n x[n] = \frac{b^2}{4} + \frac{a^2}{4} + 1 + \frac{a^2}{4} + \frac{b^2}{4} = 1 + \frac{1}{2}\left(a^2 + b^2\right)$$

Problem 5

An ideal highpass filter with linear phase has impulse response $h_d[n]$ and frequency response $H_d(e^{j\omega}) = \begin{cases} 0, & |\omega| < 0.3\pi \\ 2e^{-j\omega n_d}, & 0.3\pi < |\omega| \le \pi \end{cases}$

A linear phase FIR highpass filter with impulse response $h[n] = w_K[n]h_d[n]$ and frequency response $H(e^{j\omega})$ was obtained by multiplying $h_d[n]$ by a Kaiser window $w_K[n]$ having length $M+1 = 41$ samples and a value of β that was computed using Kaiser's design formula $M = \dfrac{A-8}{2.285\Delta\omega}$

so that the sidelobes of the magnitude spectrum of $w_K[n]$ are below 0.1 percent of the peak.

a) Determine n_d such that the resulting FIR filter has indeed linear phase;
b) Determine the impulse response $h_d[n]$ of the ideal highpass filter for the value of n_d obtained in part (a);
c) The resulting FIR highpass filter meets a set of specfications of the following form: $|H(e^{j\omega})| < \delta_1, \quad 0 \le |\omega| \le \omega_1; \quad G-\delta_2 \le |H(e^{j\omega})| \le G+\delta_2, \quad \omega_2 \le |\omega| \le \pi$
Determine the values of $\omega_1, \omega_2, \delta_1, \delta_2,$ and G.

a) For linear phase, $n_d = M/2 = 20$

b) $$h_d[n] = 2\delta[n-20] - \frac{2\sin[0.3\pi(n-20)]}{\pi(n-20)}$$

c) $$|H_d(e^{j\omega})| = \begin{cases} 0, & |\omega| < 0.3\pi \quad \text{stop band} \\ 2, & 0.3\pi < |\omega| \le \pi \quad \text{pass band} \end{cases}$$

$$A = -20\log_{10}(0.001) = 60$$

$$|H_d(e^{j\omega})|_{passband} = 2 = G \qquad \therefore\ \delta_1 = \delta_2 = 2\times 0.001 = 0.002$$

$$M = \frac{A-8}{2.285\Delta\omega} = \frac{60-8}{2.285\Delta\omega} = \frac{52}{2.285\Delta\omega} = 40 \quad \Rightarrow \quad \Delta\omega = 0.5689 = 0.181\pi$$

$$\omega_c = 0.3\pi, \quad \Delta\omega = 0.5689 = 0.181\pi,$$

$$\omega_1 = \omega_c - 0.5\Delta\omega = 0.3\pi - 0.09\pi = 0.21\pi; \quad \omega_2 = \omega_c + 0.5\Delta\omega = 0.3\pi + 0.09\pi = 0.39\pi$$

Problem 6

The system function of a stable 2nd order all-pole LTI system can be characterized by a pole-pair $re^{\pm j\theta}$ and a gain term, σ, resulting in the expression $H(z) = \sigma / A(z)$ where $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$. We have at hand two such systems $H_1(z)$ and $H_2(z)$ with $\sigma_1, r_1 e^{\pm j\theta_1}$ and $\sigma_2, r_2 e^{\pm j\theta_2}$, respectively.

a. Write the overall system function $H_a(z)$ of Fig. a and determine the scaling factor p such that it achieves unity gain for a DC signal.
b. Write the overall system function $H_b(z)$ of Fig. b and determine the scaling factor q such that it achieves unity gain for a DC signal.
c. Find $H_c(z)$ in Fig. c such that the overall system response becomes identical to $H_a(z)$.

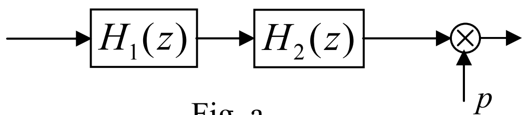


Fig. a

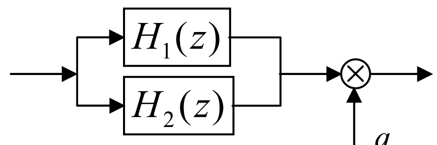


Fig. b

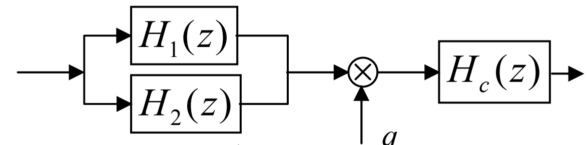


Fig. c

Answer:

a) $$H_1(z) = \frac{\sigma_1}{(1 - r_1 e^{j\theta_1} z^{-1})(1 - r_1 e^{-j\theta_1} z^{-1})} = \frac{\sigma_1}{1 - 2r_1 \cos\theta_1 z^{-1} + r_1^2 z^{-2}} \qquad H_2(z) = \frac{\sigma_2}{(1 - r_2 e^{j\theta_2} z^{-1})(1 - r_2 e^{-j\theta_2} z^{-1})} = \frac{\sigma_2}{1 - 2r_2 \cos\theta_2 z^{-1} + r_2^2 z^{-2}}$$

Let $A_1(z) = 1 - 2r_1 \cos\theta_1 z^{-1} + r_1^2 z^{-2}$ and $A_2(z) = 1 - 2r_2 \cos\theta_2 z^{-1} + r_2^2 z^{-2}$

Furthermore, $A_1 = A_1(1) = 1 - 2r_1 \cos\theta_1 + r_1^2$ and $A_2 = A_2(1) = 1 - 2r_2 \cos\theta_2 + r_2^2$

Then, $H_a(z) = p \dfrac{\sigma_1}{A_1(z)} \dfrac{\sigma_2}{A_2(z)}$ For $H_a(1) = 1$, $p\sigma_1\sigma_2 = A_1 A_2$ Thus, $p = A_1 A_2 (\sigma_1 \sigma_2)^{-1}$ and $H_a(z) = \dfrac{A_1 A_2}{A_1(z) A_2(z)}$

b) $$H_b(z) = q\left[\frac{\sigma_1}{A_1(z)} + \frac{\sigma_2}{A_2(z)}\right] = q\left[\frac{\sigma_1 A_2(z) + \sigma_2 A_1(z)}{A_1(z) A_2(z)}\right]$$

For $H_b(1) = 1$, $q\{\sigma_1 A_2 + \sigma_2 A_1\} = A_1 A_2$ Therefore, $q = \dfrac{A_1 A_2}{[\sigma_1 A_2 + \sigma_2 A_1]}$

c) $$H_b(z) = \frac{A_1 A_2}{[\sigma_1 A_2 + \sigma_2 A_1]}\left[\frac{\sigma_1 A_2(z) + \sigma_2 A_1(z)}{A_1(z) A_2(z)}\right] = \frac{A_1 A_2}{A_1(z) A_2(z)}\left[\frac{\sigma_1 A_2(z) + \sigma_2 A_1(z)}{\sigma_1 A_2 + \sigma_2 A_1}\right] = H_a(z)\left[\frac{\sigma_1 A_2(z) + \sigma_2 A_1(z)}{\sigma_1 A_2 + \sigma_2 A_1}\right]$$

$$H_c(z) = \frac{\sigma_1 A_2 + \sigma_2 A_1}{\sigma_1 A_2(z) + \sigma_2 A_1(z)}$$

Problem 7

Consider a finite-length sequence $x[n]$ of length N. Two finite-length sequences $x_1[n]$ and $x_2[n]$ of length $2N$ are constructed from according to:

$$x_1[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad x_2[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ -x[n-N], & N \le n \le 2N-1 \\ 0, & \text{otherwise} \end{cases}$$

Denote the N-point DFT of $x[n]$ as $X[k]$, and the 2N-point DFTs of $x_1[n]$ and $x_2[n]$ as $X_1[k]$ and $X_2[k]$, respectively.

a) Express $X_1[k]$ and $X_2[k]$ in terms of $X[k]$; i.e., find explicit relationship between $X[k]$ and $X_1[k]$, and $X[k]$ and $X_2[k]$;

b) Since $x[n]$ and $x_1[n]$ look identical, explain why their DFTs are different.

Answer:

$$x_1[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

$$X_1[k] = \sum_{n=0}^{2N-1} x_1[n] W_{2N}^{kn}$$

$$= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn}$$

$$= \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{n(k/2)} & \text{for even } k \\ \sum_{n=0}^{N-1} x[n] e^{-jn\pi/N} W_N^{n[(k-1)/2]} & \text{for odd } k \end{cases}$$

$$= \begin{cases} X\left[\frac{k}{2}\right] & \text{for even } k \\ X'\left[\frac{k-1}{2}\right] & \text{for odd } k \end{cases}$$

where $X'[k]$ is the N-DFT of $x[n]e^{-jn\pi/N}$

(You can continue to look at $X'[k]$ as a sampled version of $X'(e^{j\omega})$ which can be obtained from convolving $X(e^{j\omega})$ with the DT-FT of $e^{-jn\pi/N}$. No need to carry this all the way through to get full credit, but the concept of zero-padding needs to be understood.)

$$x_2[n] = x_1[n] * h[n]$$

where

$$h[n] = \begin{cases} 1, & n = 0 \\ -1, & n = N \\ 0, & \text{elsewhere} \end{cases}$$

$2N$ – DFT of $h[n]$ is $H[k] = \sum_{n=0}^{2N-1} h[n] W_{2N}^{kn} = W_{2N}^{k0} - W_{2N}^{kN}$

$$= 1 - e^{-jk\pi} = 1 - (-1)^k$$

$$X_2[k] = X_1[k] H[k]$$

$$= \begin{cases} 0 & \text{for even } k \\ 2X_1[k] & \text{for odd } k \end{cases}$$