Available online at www.sciencedirect.com

SCIENCE DIRECT®

Statistics & Probability Letters 72 (2005) 71–81

STATISTICS & PROBABILITY LETTERS

www.elsevier.com/locate/stapro

Minimax correlation between a line segment and a beamlet

Xiaoming Huo[1]

School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0205, USA

Received 12 March 2004; received in revised form 7 December 2004; accepted 20 December 2004
Available online 28 January 2005

Abstract

The minimax correlation between a line segment and a beamlet is proved to be $2^{-3/4}$. The result is useful for design of efficient line-segment detectors with noisy data.

Keywords: Multiscale geometric detection; Beamlets; ε-net; Detecting line segments in noisy data

1. Introduction

In developing detectors for line segments of unknown length, location, and orientation in noisy two-dimensional imagery, one is faced immediately with the problem that there are $\sim n^4/2$ line segments in an $n \times n$ image. To make practical algorithms, Donoho and Huo (2001) made the suggestion of considering a specially chosen family of line segments, which were called *beamlets*. This family is dyadically organized, containing line segments of length approximately $n/(2^\ell)$ pixels, for each integral ℓ between 0 and $\log_2(n)$. Arias-Castro et al. (2003) developed a formal analysis of line segment detection. Their work shows implicitly that a key issue in developing a fast algorithm is the following geometric problem.

Let B and S denote two line segments in a unit square $[0,1)^2$; the *correlation* between them is defined as

$$\rho(B,S) = \frac{|B \cap S|}{\sqrt{|B| \cdot |S|}},$$

E-mail address: xiaoming@isye.gatech.edu (X. Huo).
[1]This work has been partially supported by National Science Foundation DMS-0140587 and 0346307.

0167-7152/$ - see front matter
doi:10.1016/j.spl.2004.12.004

where $|B|, |S|$, and $|B \cap S|$ denote the lengths of segments B, S, and their intersection $B \cap S$, respectively. If segments B and S are not on the same line, set $|B \cap S| = 0$. Let now S range over a general family of line segments and B over a restricted family of line segments. The quantity

$$\min_{S} \max_{B} \rho(B, S)$$

measures the ability of the restricted family of B's to detect the presence of S's.

Theorem 1.1. *Let $\mathbb{S}$ denote the family of all line segment in $[0,1]^2$, and $\mathbb{B}$ the family of beamlets (defined below). Then*

$$\min_{S \in \mathbb{S}} \max_{B \in \mathbb{B}} \rho(B, S) = 2^{-3/4} (\approx 0.5946). \tag{1}$$

Throughout this paper, S stands for a line segment and B stands for a beamlet.

This result, working from a continuum viewpoint (S and B are line segments in the square), shows that algorithms may be developed in which order $n^2 \log(n)$ discrete beamlets can detect any of $n^4/2$ line segments with a sacrifice of at most $2^{3/4}$ in detection threshold. Compare Arias-Castro et al. (2003).

We consider a continuum beamlet dictionary defined geometrically as follows. Compare Donoho and Huo (2001), Donoho (1999) and Huo (1999).

- In the first step, the unit square is dyadically partitioned into 4 dyadic sub-squares. Each sub-square is then equally divided into 2×2 smaller dyadic sub-squares. This process is repeated.
- Associated to each square, consider all the line segments B with endpoints on the boundary of the squares.

Fig. 1 gives an illustration of beamlets at different scales.

The quantity $\rho(B, S)$ can be viewed as the limit of a correlation of two regions. Let $\mathbf{1}_{B,\delta}$ and $\mathbf{1}_{S,\delta}$ ($\delta > 0$) denote the indicator functions of the δ-neighborhood of the line segments B and S, respectively. Then

$$\rho(B, S) = \lim_{\delta \to 0} \frac{\int \mathbf{1}_{B,\delta} \cdot \mathbf{1}_{S,\delta}}{\sqrt{\int \mathbf{1}_{B,\delta}} \sqrt{\int \mathbf{1}_{S,\delta}}}.$$

The rest of the paper is organized as follows. The proof of the main theorem is presented in Section 2. Some discussion and concluding marks are provided in Section 3.

2. Proof

To simplify the proof, we define

$$G(S) = \max_{B} \frac{|B \cap S|}{\sqrt{|B| \cdot |S|}}. \tag{2}$$

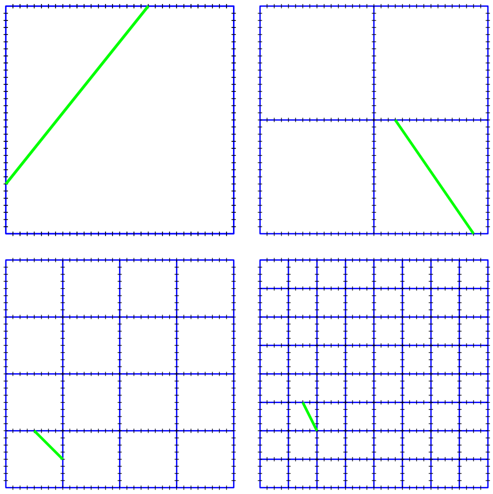

Fig. 1. Dyadic squares marked with vertices and several beamlets.

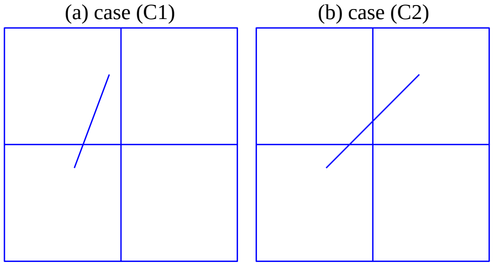

Fig. 2. Two cases for line segment S.

We consider a dyadic partition of a unit square. At each stage, there is a vertical partition line and a horizontal partition line. Without loss of generality, assume that segment S intersects *at least* one of the two first-level dyadic partition lines; otherwise, the line segment lives in a dyadic subsquare, which we can treat as $[0,1]^2$ and start over. We have the following possible cases:

(C1) segment S intersects only one partition line;
(C2) segment S intersects both the horizontal and the vertical partition lines.

The above two cases are illustrated in Fig. 2(a) and (b).

Consider case (C1) first. The next level partition is illustrated in Fig. 3. Without loss of generality, assume that S intersects the line segment CA_{12}; otherwise, it can be treated as (C2). Consider the following three subcases:

(C1-1) the other end of segment S ends in subsquare $B_{11}B_{12}DA_{21}$;

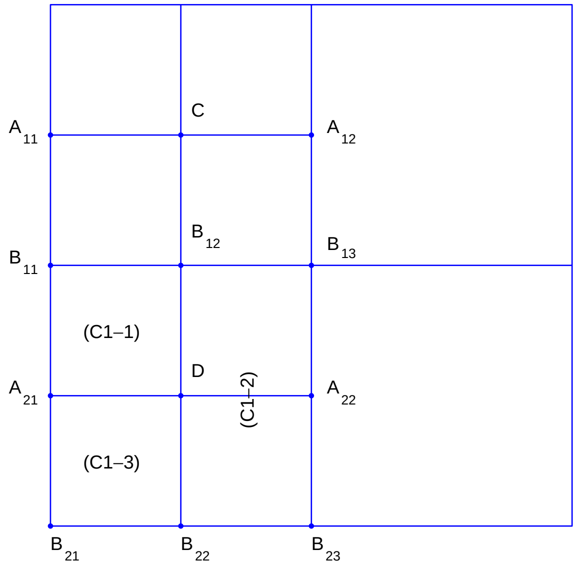

Fig. 3. Notations for case (C1).

(C1-2) the other end of segment S ends in rectangle $B_{12}B_{13}B_{23}B_{22}$;
(C1-3) the other end of segment S ends in subsquare $A_{21}DB_{22}B_{21}$.

For case (C1-3), take a beamlet B at the coarsest scale covering S. This can be done by extending S to the boundary of the largest square. It is not hard to see that the length of S is at least half of the length of beamlet B. This can be seen by projecting both S and B onto the straight line $A_{12}B_{23}$. Hence

$$\rho(B,S) = \frac{|B \cap S|}{\sqrt{|B||S|}} = \sqrt{\frac{|S|}{|B|}} \geqslant \sqrt{1/2}. \tag{3}$$

This is consistent with (1).

In case (C1-2), consider the projection of S on the line $A_{12}B_{23}$. If S crosses DA_{22}, choose B by extending S to the boundary of the largest square. Similar to the previous case, the S is at least as half long as B. Hence we have (3).

Again in case (C1-2), if S does *not* cross line DA_{22}, we develop a method which will be used in future discussion of other cases. Assume the length of segment $A_{12}B_{13}$ is 1. Note the scaling. Let β and α be the lengths of the projections of S in interval $B_{13}A_{22}$ and above point A_{12}, respectively. See Fig. 4 for an illustration. We can verify:

$$G(S) = \max\left\{ \frac{1+\alpha}{\sqrt{2\cdot(1+\alpha+\beta)}}, \frac{1+\alpha+\beta}{\sqrt{4\cdot(1+\alpha+\beta)}}, \frac{1}{\sqrt{1\cdot(1+\alpha+\beta)}} \right\}, \tag{4}$$

where $0<\alpha,\beta<1$ are determined by the projection of segment S, and

$$\min_S G(S) = \min_{\alpha,\beta} \{\text{the right-hand side of (4)}\}. \tag{5}$$

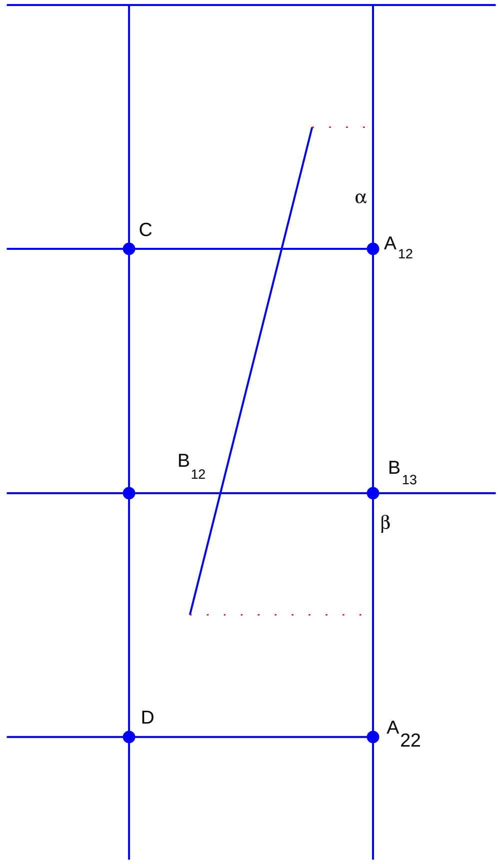

Fig. 4. Additional notation for case (C1-2).

The three terms inside the brackets on the right-hand side of (4) are obtained $\rho(B, S)$ by choosing different beamlets B. We have the following result.

Lemma 2.1. *The function in* (5) *has minimum value* $2^{-1/2}$.

Proof. Let I–III denote the three terms inside the brackets on the right-hand side of (4), the regions of (α, β) in which a term achieves maximal is depicted in Fig. 5, where the three partition lines have forms: $\alpha = \sqrt{2} - 1, \alpha + \beta = 1$, and $\beta = (\sqrt{2} - 1)(1 + \alpha)$. They intersect at a common point $(\alpha, \beta) = (\sqrt{2} - 1, 2 - \sqrt{2})$. Let $V(\mathrm{I})$, $V(\mathrm{II})$, and $V(\mathrm{III})$ denote the value of $G(S)$ in the above regions. We have

$$V(\mathrm{I}) \geqslant \frac{1 + \alpha}{\sqrt{2}\sqrt{\sqrt{2}(1 + \alpha)}} = \frac{\sqrt{1 + \alpha}}{2^{3/4}} \geqslant \frac{\sqrt{\sqrt{2}}}{2^{3/4}} = 2^{-1/2},$$

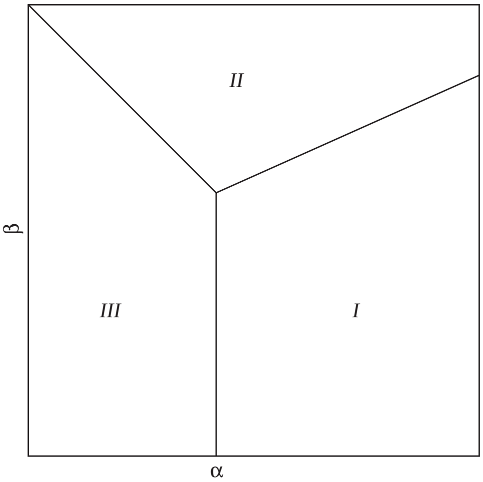

Fig. 5. Partition of (α, β) plane for the proof of Lemma 2.1.

$$V(\mathrm{II}) \geqslant \frac{1}{2}\sqrt{1+\alpha+\beta} \geqslant 1/\sqrt{2} = 2^{-1/2} \quad \text{and}$$
$$V(\mathrm{III}) \geqslant 1/\sqrt{2} = 2^{-1/2}.$$

These proved the lemma. □

We have so far proved the theorem in case (C1-2).

The consideration for case (C1-1) is more complex. It is divided into two cases.

(C1-1-1) S traverses line segment $B_{12}B_{13}$;
(C1-1-2) S traverses line segment $B_{11}B_{12}$.

In case (C1-1-1), by projecting to line $A_{12}B_{23}$, one can see that the beamlet in the square $CA_{12}B_{13}B_{12}$ is at least $1/3$ of the length of S. To simplify the argument, we scale the projection of the beamlet in $CA_{12}B_{13}B_{12}$ to length 1, and apply the same scaling to all other projections. Apparently, the projection of S is shorter than 3. There are two possibilities:

(1) If the projection of S is no more than $2\sqrt{2}$, set B to be the beamlet in $CA_{12}B_{13}B_{12}$. Then

$$\rho(B,S) = \sqrt{|B|/|S|} \geqslant \sqrt{1/(2\sqrt{2})} = 2^{-3/4}.$$

Hence $G(S) \geqslant 2^{-3/4}$.

(2) If $|S| > 2\sqrt{2}$, choose the beamlet B in the square having bottom $B_{11}B_{13}$ and midpoints A_{12} and A_{11} (on left and right sides),

$$\rho(B,S) = \frac{|B \cap S|}{\sqrt{|B||S|}} \geqslant \frac{2\sqrt{2}-1}{\sqrt{2 \cdot 3}} = \frac{2\sqrt{2}-1}{\sqrt{6}} (\approx 0.7465) > 2^{-3/4}.$$

The theorem is proved in the case of (C1-1-1).

For case (C1-1-2), a parameterization is introduced in Fig. 6. The notations α, β, and γ stand for the percentages of the projection of S in various intervals, $0 \leqslant \alpha, \beta, \gamma \leqslant 1$. One has

$$G(S) = \max\left\{ \frac{\max\{\alpha/\sqrt{\alpha}, (1-\alpha)/\sqrt{1-\alpha}, \beta, \gamma\}}{\sqrt{1+\beta+\gamma}}, \frac{1+\beta}{\sqrt{1+\beta+\gamma}\sqrt{2}}, \frac{1+\beta+\gamma}{\sqrt{1+\beta+\gamma}\cdot 2} \right\}, \tag{6}$$

where the six terms on the right-hand side of the above equation correspond to the cases when $|S| = 1+\alpha+\beta$, $|B| = \alpha, 1-\alpha, 1, 1, 2, 4$, and $|B \cap S| = \alpha, 1-\alpha, \beta, \gamma, 1+\beta, 1+\alpha+\beta$, respectively. Again details are omitted.

Lemma 2.2. *With $G(S)$ defined in* (6), $G(S) \geqslant 2^{-3/4}$.

Proof. We will use:

$$\max\{\sqrt{\alpha}, \sqrt{1-\alpha}\} \geqslant 1/\sqrt{2}.$$

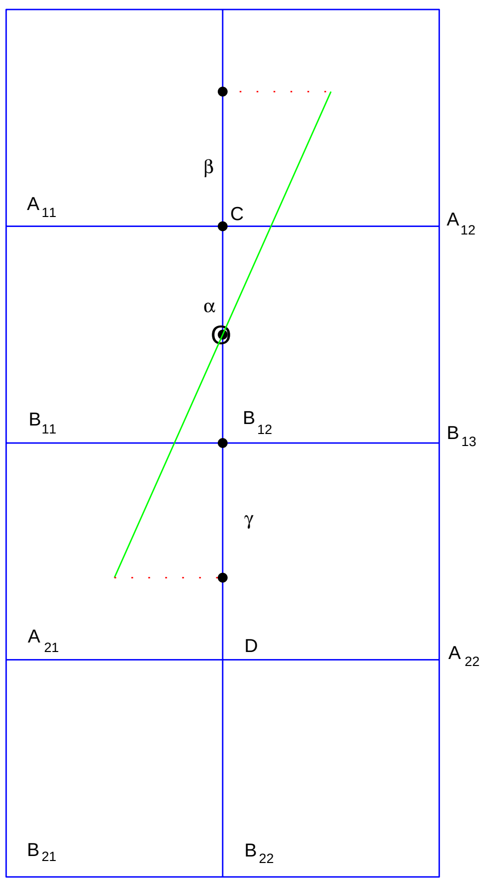

Fig. 6. Parameterization in case (C1-1-2). When $\beta = 0, \alpha = 1/2$, and $\gamma = \sqrt{2} - 1$, the minimax correlation $2^{-3/4}$ is achieved.

We also need:

$$\frac{1+\beta}{\sqrt{2}} \geqslant \frac{1}{\sqrt{2}} \quad \text{and} \quad \frac{1+\beta+\gamma}{2} \geqslant \max\{\beta,\gamma\}.$$

We have

$$G(S) \geqslant \frac{1}{\sqrt{1+\beta+\gamma}} \max\left\{\frac{1+\beta}{\sqrt{2}}, \frac{1+\beta+\gamma}{2}\right\}. \tag{7}$$

Consider two possibilities:

(1) When $(\sqrt{2}-1)(1+\beta) \geqslant \gamma$, the first term on the right-hand side of (7) is large. We have

$$G(S) \geqslant \frac{1+\beta}{\sqrt{(1+\beta)\sqrt{2}\sqrt{2}}} \geqslant 2^{-3/4}.$$

(2) When $(\sqrt{2}-1)(1+\beta) \leqslant \gamma$, we have

$$G(S) \geqslant \frac{\sqrt{1+\beta+\gamma}}{2} \geqslant \frac{\sqrt{\sqrt{2}(1+\beta)}}{2} \geqslant 2^{-3/4}.$$

In both cases, the equality is achieved when $\beta = 0, \gamma = \sqrt{2}-1$, and $\alpha = \frac{1}{2}$. □

So case (C1-1-2) is proved.

Now consider case (C2). See the labelling in Fig. 7. Due to the symmetry of the problem, without loss of generality, one can assume that S traverses the line segment $A_{11}A_{12}$. Consider the following subcases.

(C2-1) S goes below line $B_{21}B_{23}$;
(C2-2) S ends in the square $B_{12}B_{13}B_{23}B_{22}$;
(C2-3) S ends in the square $B_{11}B_{12}B_{22}B_{21}$.

In case (C2-1), take a beamlet B by extending S to global scale. Similar to a previous case, one can show that $|S| \geqslant |B|/2$. This will lead to the observation that for any S in this category, $G(S) \geqslant 1/\sqrt{2}$.

Case (C2-2) is very similar to case (C1-1-2). Given the notation in Fig. 8, and $0<\alpha,\beta,\gamma<1$, we have

$$G(S) = \frac{1}{\sqrt{1+\beta+\gamma}} \max\left\{\sqrt{1-\alpha}, \sqrt{\alpha}, \gamma, \beta, \frac{\alpha+\beta}{\sqrt{1+\alpha}}, \frac{\gamma}{\sqrt{2}}, \frac{1+\beta+\gamma}{2}\right\}. \tag{8}$$

The above is based on the cases when $|S| = 1+\beta+\gamma$, $|S \cap B| = 1-\alpha, \alpha, \gamma, \beta, \alpha+\beta, \gamma, 1+\beta+\gamma$, and $|B| = 1-\alpha, \alpha, 1, 1, 1+\alpha, 2, 4$. Details are again omitted.

Lemma 2.3. *For* $G(S)$ *given in* (8), $G(S) \geqslant 2^{-3/4}$.

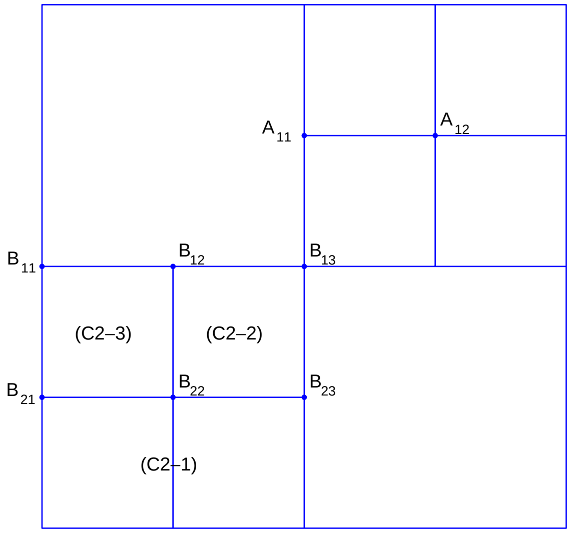

Fig. 7. Notations for case (C2).

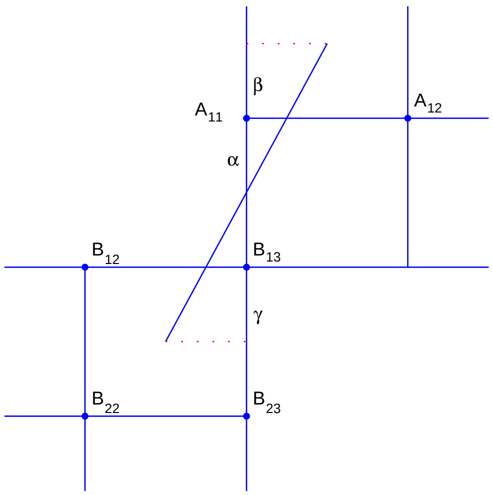

Fig. 8. Notations for case (C2-2).

Proof. Clearly,

$$\max\left\{\beta, \gamma, \frac{1+\beta+\gamma}{2}\right\} = \frac{1+\beta+\gamma}{2}.$$

Secondly,

$$\max\left\{\sqrt{1-\alpha}, \sqrt{\alpha}\right\} \geqslant 1/\sqrt{2} \quad \text{and} \quad \max\{\gamma/\sqrt{2}, 1/\sqrt{2}\} = 1/\sqrt{2}.$$

Combining these, one can conclude

$$
\begin{aligned}
G(S) &\geqslant \frac{1}{\sqrt{1+\beta+\gamma}} \max\left\{\frac{1}{\sqrt{2}}, \frac{1+\beta+\gamma}{2}, \frac{\alpha+\beta}{\sqrt{1+\alpha}}\right\} \\
&\geqslant \max\left\{\frac{1}{\sqrt{2}\cdot\sqrt{1+\beta+\gamma}}, \frac{\sqrt{1+\beta+\gamma}}{2}\right\} \\
&\geqslant 2^{-3/4}. \qquad \square
\end{aligned}
$$

We have proved case (C2-2).

For case (C2-3), consider the projections on line $A_{11}B_{23}$, and scale the length $|A_{11}B_{13}|$ to 1. Consider two possibilities:

(1) If the length of the projection of S is at least $\sqrt{2}$ (after scaling), take a beamlet B extending S to the coarsest scale, and then

$$G(S) \geqslant \sqrt{\frac{|S|}{|B|}} \geqslant \sqrt{\frac{\sqrt{2}}{4}} = 2^{-3/4}.$$

(2) If the length of the projection of S is no larger than $\sqrt{2}$, take B as the longer one of the two beamlets between line $A_{11}A_{12}$ and line $B_{12}B_{13}$. The projection of B must be at least $\frac{1}{2}$. We have

$$G(S) \geqslant \sqrt{\frac{|B|}{|S|}} \geqslant \sqrt{\frac{1/2}{\sqrt{2}}} = 2^{-3/4}.$$

We have now proved Theorem 1.1.

3. Discussion and conclusion

Discussion. A related minimax quantity, equal to $\frac{1}{7}$, was derived in Huo (2005). The difference is that in the present work, we do *not* require beamlet B to be a subset of the line segment S (i.e. $B \subset S$). This relaxation in condition leads to a substantially better minimax value.

Conclusion. We proved that the minimax correlation between a line segment and a beamlet is $2^{-3/4}$. This constant is motivated by a particular problem in multiscale geometric detection (MGD): more specifically, detect line segments via beamlets. The exact minimax value is useful in designing fast algorithms with high detection sensitivity.

Acknowledgements

The comments from an anonymous referee helps greatly in improving the presentation of this paper.

References

Arias-Castro, E., Donoho, D.L., Huo, X., 2003. Near-optimal detection of geometric objects by fast multiscale methods. Submitted. http://www-stat.stanford.edu/~donoho/Reports/2003/MultiScaleDetect.pdf.

Donoho, D.L., 1999. Wedgelets: nearly minimax estimation of edges. Ann. Statist. 27 (3), 859–897.

Huo, X., August 1999. Sparse image representation via combined transforms. Ph.D. thesis, Stanford University.

Donoho, D., Huo, X., 2001. Beamlets and Multiscale Image Analysis, Lecture Notes in Computational Science and Engineering, vol. 20, Springer, Berlin, pp. 149–196.

Huo, X., 2005. Exact lower bound for proportion of maximally embedded beamlet. Appl. Math. Lett., to appear.