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Statistical interpretation of the importance of phase information in signal and image reconstruction

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Abstract

In the Fourier representation of signals and images, phases have long been realized to be more important than magnitudes in the reconstruction. In this paper, a justification is presented from a statistical viewpoint. The main result shows that under random magnitudes, the DC component of the inverse Fourier transform converges to a positive value, while all the other components converge to zero. For random phases, such a result does not exist.

Keywords: Reconstruction; Fourier transform; Importance of phases

1. Introduction

For both one dimensional and multi-dimensional signals, the magnitudes and the phases of the Fourier transform are playing different roles in the reconstruction. In general, important features of a signal are preserved in phases. The realization of this "importance of phase" has a long history. In the signal processing literature, many experiments were performed. Oppenheim, Lim, et al. are in the first group who emphasized this phenomenon (Oppenheim and Lim, 1981; Oppenheim et al., 1979). They used numerical experiments to illustrate the similarity between a signal and its phase-kept reconstruction.

A typical experiment runs as follows. Given two images A and B, one can compute their 2-D Fourier transforms $\mathscr{F}(A)$ and $\mathscr{F}(B)$. Based on $\mathscr{F}(A)$ and $\mathscr{F}(B)$, two image reconstructions are conducted. One (denoted by R_1) is the inverse Fourier transform of the combination of the phases of $\mathscr{F}(A)$ and the magnitudes of $\mathscr{F}(B)$. The other (denoted by R_2) is the inverse Fourier transform of the combination of the phases of $\mathscr{F}(B)$ and the magnitudes of $\mathscr{F}(A)$. (Note the original images A and B can be considered as *real* matrices. But there is no guarantee that R_1 and R_2 are real. One can overcome this by taking the magnitudes or the real parts of R_1 and R_2.) It is found that in nearly all the times, reconstruction R_1 is similar to image A, while reconstruction R_2 is similar to image B. A natural interpretation of this phenomenon is that in the image (or signal) reconstruction, phase information is more important than magnitude information.

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This paper addresses the above problem from a statistical point of view. We show that if the phases are randomly re-assigned, the reconstructed signal is likely to be severely distorted. At the same time, if the magnitudes are randomly re-assigned, the distortion is automatically controlled within a region, whose size is given by the special structure of the discrete Fourier transform together with the distribution of the signal. More specifically, the reconstructed signal from the original phases and the re-assigned magnitudes is close to a convolution of the original signal and a Dirac-like signal.

The rest of this paper is organized as follows. Section 2 illustrates the main results with two simulations, one on 1-D signals (Section 2.1) and the other on 2-D images (Section 2.2). Section 3 develops our main theoretical results. Section 4 reviews some other interpretation, and summarizes some key results regarding computing, which is another important problem in reconstruction. Section 5 makes some concluding remarks.

2. Illustration

2.1. 1-D signal

The following numerical experiment illustrates our major points. We first generate a complex sequence ($\hat{z}_1$) with all the magnitudes being equal to one, but the phases being random and uniformly distributed between $[-\pi, \pi]$, i.e., the Fourier transform of an *all pass filter*. We find that the inverse Fourier transform of $\hat{z}_1$ is random, with its magnitudes presented in Fig. 1 (top left) and phases presented in Fig. 1 (top right).

Next, we generate another sequence ($\hat{z}_2$) with zero phases, random magnitudes, and the same ℓ_2 norm as $\hat{z}_1$. The inverse Fourier transform of $\hat{z}_2$ is computed. This time, the first element (i.e., the DC component) of the inverted Fourier transform consistently and significantly has a larger magnitude than the rest elements. See the magnitudes (resp. phases) of the reconstructed signal in Fig. 1 bottom left (resp. right).

The implication of this experiment is that a random distortion of the phases can severely distort the original signal, while a random distortion of the magnitudes tends to introduce less distortion in the reconstructed signal, because the convolved sequence is close to a Dirac sequence, which is a sequence taking 1 in one position and zero elsewhere.

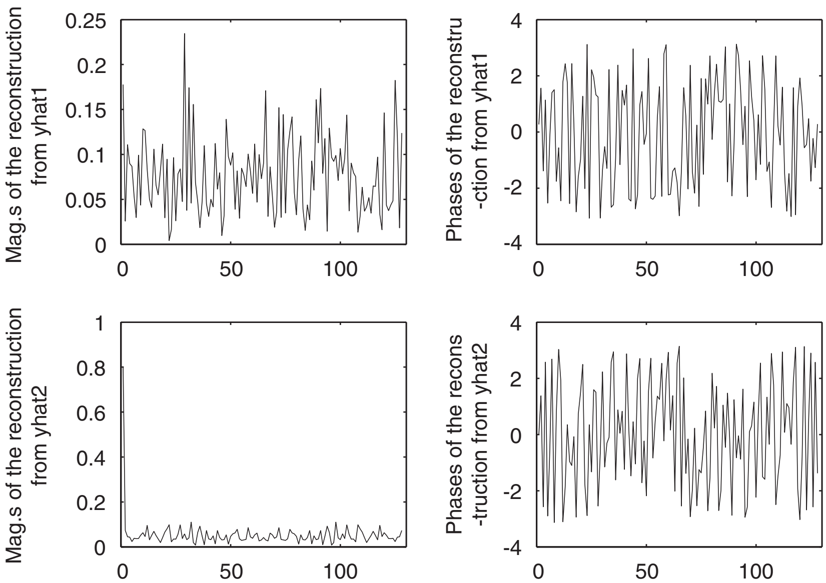

Fig. 1. An illustration for 1-D signal. The top (resp., bottom) row contains the magnitudes and the phases of the reconstructed signal from a 1-D data with random phases (resp., magnitudes).

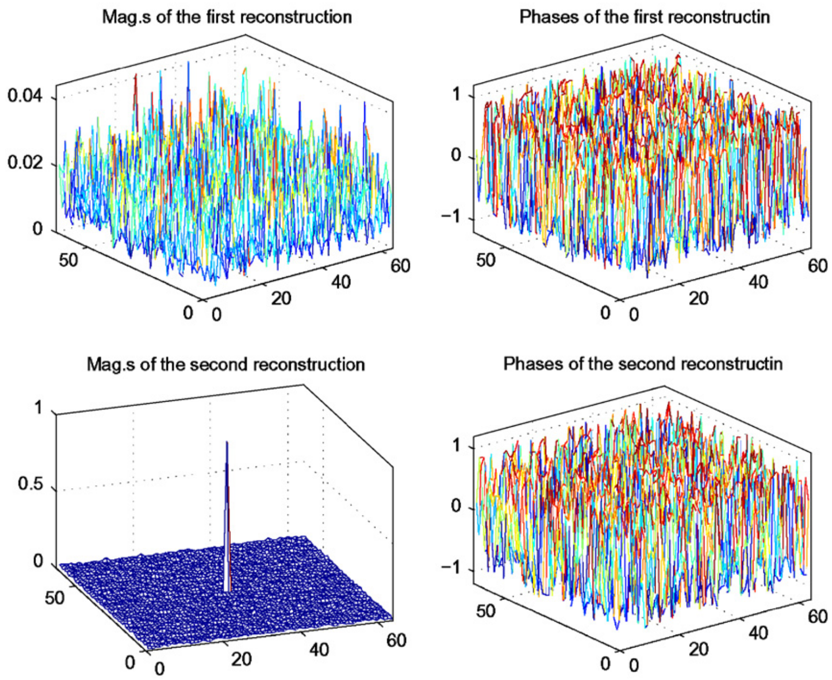

Fig. 2. An illustration for 2-D data. The top (resp., bottom) row contains the magnitudes and the phases of the reconstructed image from a 2-D data with random phases (resp., magnitudes).

2.2. 2-D image

The same phenomenon can be observed in 2-D. If we change the sequences in the previous experiment into 2-D arrays with the same constraints and randomization, an identical result can be observed (shown in Fig. 2). The reconstructed signal from random phases still appears random (Fig. 2, top row), while the reconstructed signal from random magnitudes has a dominant magnitude at the DC component (Fig. 2, bottom row).

3. Main result

For simplicity, we consider the signal (i.e., 1-D) reconstruction, with a remark that all of the results in this paper can be easily extended to an arbitrary dimension. The generalization is based on a simple fact: a high dimensional Fourier transform is a tensor-product of 1-D Fourier transforms.

We start with notations. Let n be a positive integer, $x = \{x_0, x_1, \ldots, x_{n-1}\}$ denote a sequence, and $\widehat{x} = \{\widehat{x}_0, \widehat{x}_1, \ldots, \widehat{x}_{n-1}\}$ denote the discrete Fourier transform of x, i.e.,

$$\widehat{x}_j = \sum_{i=0}^{n-1} x_i \omega^{ij} / \sqrt{n}, \quad j = 0, 1, \ldots, n-1,$$

where $\omega = \exp\{-(2\pi/n)\sqrt{-1}\}$. The following is the inverse Fourier transform:

$$x_i = \sum_{j=0}^{n-1} \widehat{x}_j \omega^{-ij} / \sqrt{n}, \quad i = 0, 1, 2, \ldots, n-1.$$

Let $m(\widehat{x}_j)$ and $\phi(\widehat{x}_j)$ denote the magnitude and the phase of the complex number $\widehat{x}_j$, respectively. We have $\widehat{x}_j = m(\widehat{x}_j) \exp\{\phi(\widehat{x}_j)\sqrt{-1}\}$.

Suppose another sequence $\widetilde{x} = \{\widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_{n-1}\}$ has its Fourier transform as follows:

$$\mathscr{F}(\{\widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_{n-1}\}) = \widehat{\widetilde{x}} = \{\widehat{x}_0\widehat{y}_0, \widehat{x}_1\widehat{y}_1, \ldots, \widehat{x}_{n-1}\widehat{y}_{n-1}\},$$

where $\widehat{y} = \{\widehat{y}_0, \widehat{y}_1, \ldots, \widehat{y}_{n-1}\}$ is a new sequence of complex numbers. Let $y = \{y_0, y_1, \ldots, y_{n-1}\}$ denote the inverse Fourier transform of $\widehat{y}$. Using a well-known fact of the discrete Fourier transform, we know that $\widetilde{x}$ is a cyclic convolution of sequences x and y:

$$\widetilde{x}_k = \sum_{i_1+i_2=k \text{ or } n+k} x_{i_1} y_{i_2}, \quad 0 \leqslant k \leqslant n-1. \tag{3.1}$$

One can treat $\widehat{\widetilde{x}}$ as a distorted version of $\widehat{x}$ (distorted by $\widehat{y}$). By doing inverse Fourier transform on $\widehat{\widetilde{x}}$, we hope that the reconstructed signal $\widetilde{x}$ is close to the original signal x. In the magnitude-based reconstruction, where all the magnitudes of $\widehat{x}$ are kept unchanged, the phases are distorted: i.e., one imposes $m(\widehat{y}_j) = 1, \forall j$, and randomly specifies $\phi(\widehat{y}_j)$'s. Similarly, in the phase-based reconstruction, one imposes $\phi(\widehat{y}_j) = 0, \forall j$, and randomly specifies $m(\widehat{y}_j)$'s. From (3.1), it is obvious that the sequence $\widetilde{x}$ is close to the original x if sequence y is close to a Dirac signal. Whether or not sequence y is close to a Dirac is the key problem. Readers may review the previous simulations to gain more intuition.

The following theorems justify what we observed in the simulations.

Theorem 3.1. *Consider a set of positive magnitudes* $\rho = \{\rho_1, \rho_2, \ldots, \rho_n\}$, *where* $\rho_i > 0$, $1 \leqslant i \leqslant n$. *Let* $\pi_1 = \{\pi_1^1, \pi_2^1, \ldots, \pi_n^1\}$ *and* $\pi_2 = \{\pi_1^2, \pi_2^2, \ldots, \pi_n^2\}$ *be two random permutations of* $\{1, 2, \ldots, n\}$. *Assign* $s_i = \rho_{\pi_i^1}/\rho_{\pi_i^2}$. *We have the following*:

(1) $(1/n)\sum_{i=1}^n s_i \geqslant 1$, *and*

(2) *for set* $\varepsilon_n = \{\varepsilon_{1,n}, \varepsilon_{2,n}, \ldots, \varepsilon_{n,n}\}$, $\varepsilon_{i,n} \in \mathscr{R}$, $1 \leqslant i \leqslant n$, *satisfying* $\sum_{i=1}^n \varepsilon_{i,n} = 0$ *and* $\sum_{i=1}^n \varepsilon_{i,n}^2 = n$, *if* $\mathbf{Var}(s_i)/n \to 0$, *we have*

$$\frac{1}{n}\sum_{i=1}^n \varepsilon_{i,n} s_i \longrightarrow 0, \quad \textit{in probability, as } n \to \infty.$$

In this theorem, s is certain distortion applied on the magnitudes ρ. It is chosen so that s_i's are identically distributed. Result (1) indicates that the DC component of the inverse Fourier transform of s is no less than 1. Result (2) indicates that all the other components are small and converge to zero in probability. The proof is in the following.

Proof. (1) This result only requires a simple fact: the arithmetic mean is no less than the geometric mean, we omit the details.

(2) This part will take more efforts. Firstly, noting that $\mathbf{E}(s_i)$ is a constant, we have

$$\mathbf{E}\left(\frac{1}{n}\sum_{i=1}^n \varepsilon_{i,n} s_i\right) = \frac{1}{n}\sum_{i=1}^n \varepsilon_{i,n}\mathbf{E}(s_i) = 0. \tag{3.2}$$

Secondly, we can verify the following results:

(a) We have

$$\begin{aligned}\mathbf{Var}(s_i) &= \mathbf{E}(s_i^2) - [\mathbf{E}(s_i)]^2 \\ &= \frac{1}{n^2}\left(\sum \rho_i^2\right)\left(\sum \frac{1}{\rho_i^2}\right) - \frac{1}{n^4}\left(\sum \rho_i\right)^2\left(\sum \frac{1}{\rho_i}\right)^2.\end{aligned} \tag{3.3}$$

Here, "$\sum$" stands for "$\sum_{i=1}^n$". The same convention applies to the rest of the proof.

(b) For $i \neq j$,

$$\begin{aligned}\mathbf{Cov}(s_i, s_j) &= \mathbf{E}(s_i s_j) - \mathbf{E}(s_i)\mathbf{E}(s_j) \\ &= \mathbf{E}(s_i s_j) - \frac{1}{n^4}\left(\sum \rho_i\right)^2\left(\sum \frac{1}{\rho_i}\right)^2.\end{aligned} \tag{3.4}$$

(c) Denote $S = \sum \rho_i$ and $I = \sum 1/\rho_i$, we have

$$\begin{aligned}\mathbf{E}(s_i s_j) &= \frac{1}{[n(n-1)]^2} \sum \rho_i (S - \rho_i) \sum \frac{1}{\rho_i}\left(I - \frac{1}{\rho_i}\right) \\ &= \frac{1}{n^2(n-1)^2}\left[S^2 - \sum \rho_i^2\right]\left[I^2 - \sum \frac{1}{\rho_i^2}\right] \\ &= \frac{1}{n^2(n-1)^2} S^2 I^2 + \frac{1}{n^2(n-1)^2}\sum \rho_i^2 \sum \frac{1}{\rho_i^2} - \frac{1}{n^2(n-1)^2} S^2 \sum \frac{1}{\rho_i^2} - \frac{1}{n^2(n-1)^2} I^2 \sum \rho_i^2.\end{aligned}$$

Because $n \cdot \sum 1/\rho_i^2 \geqslant I^2$ and $n \cdot \sum \rho_i^2 \geqslant S^2$, we continue for $\mathbf{E}(s_i s_j)$:

$$\mathbf{E}(s_i s_j) \leqslant \frac{1}{n^2(n-1)^2}\sum \rho_i^2 \sum \frac{1}{\rho_i^2} + \frac{n-2}{n^3(n-1)^2} S^2 I^2. \tag{3.5}$$

(d) Combing (3.4) and (3.5), we have

$$\begin{aligned}\mathbf{Cov}(s_i, s_j) &\leqslant \frac{1}{n^2(n-1)^2}\sum \rho_i^2 \sum \frac{1}{\rho_i^2} - \frac{1}{n^4(n-1)^2} S^2 I^2 \\ &\overset{\text{by (3.3)}}{=} \frac{1}{(n-1)^2}\mathbf{Var}(s_i).\end{aligned} \tag{3.6}$$

Finally, regarding $\mathbf{Var}((1/n)\sum \varepsilon_{i,n} s_i)$, we have

$$\begin{aligned}\mathbf{Var}\left(\frac{1}{n}\sum \varepsilon_{i,n} s_i\right) &= \frac{1}{n^2}\left[\sum \varepsilon_{i,n}^2 \mathbf{Var}(s_i) + \sum_{i \neq j} \varepsilon_{i,n}\varepsilon_{j,n}\mathbf{Cov}(s_i, s_j)\right] \\ &\leqslant \frac{\mathbf{Var}(s_i)}{n^2}\left[n + \sum_{i \neq j}\frac{\varepsilon_{i,n}\varepsilon_{j,n}}{(n-1)^2}\right].\end{aligned}$$

Because

$$\sum_{i \neq j} \varepsilon_{i,n}\varepsilon_{j,n} < \left(\sum |\varepsilon_{i,n}|\right)^2 \leqslant n \cdot \sum \varepsilon_{i,n}^2 = n^2,$$

we have

$$\begin{aligned}\mathbf{Var}\left(\frac{1}{n}\sum \varepsilon_{i,n} s_i\right) &< \frac{\mathbf{Var}(s_i)}{n^2}\left[n + \frac{n^2}{(n-1)^2}\right] \\ &< \frac{2}{n}\mathbf{Var}(s_i) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

The last inequality is true when n is large enough. The above proves (2). □

Remark. It is necessary to assume that $\mathbf{Var}(s_i)/n \to 0$ as $n \to \infty$. Readers can verify that if $\rho_i = 2^{i-1}, 1 \leqslant i \leqslant n$, we will have $\mathbf{Var}(s_i) \asymp \frac{4}{9}(4^n/n^2)$, which leads to $\mathbf{Var}(s_i)/n \to \infty$, as $n \to \infty$. However, if the magnitude ρ_i's are bounded, $\mathbf{Var}(s_i)/n$ will converge to zero, which is the following proposition.

Proposition 3.2. *If* $0 < m \leqslant \rho_i < M \leqslant \infty$, $i = 1, 2, \ldots, n$, *then* $(1/n)\mathbf{Var}(s_i) \longrightarrow 0$, *as* $n \to \infty$.

Proof. The following is standard:

$$\begin{aligned}\mathbf{Var}(s_i) &\overset{\text{by (3.3)}}{=} \frac{1}{n^2}\left(\sum \rho_i^2\right)\left(\sum \frac{1}{\rho_i^2}\right) - [\mathbf{E}(s_i)]^2 \\ &\leqslant \frac{1}{n^2}\left(n \cdot M^2\right)\left(n \cdot \frac{1}{m^2}\right) = \frac{M^2}{m^2} < \infty. \qquad \square\end{aligned}$$

Note that in reality, the above condition can be easily satisfied.

The above results show that randomly permuting magnitudes have nearly the same effect of convolving with a Dirac-like sequence, which makes the reconstructed signal close to the original. Based on the proofs, we can find that one sufficient condition of getting a Dirac-like inverse Fourier transform is to satisfy the following: (a) s_i's are identically distributed; (b) $\mathbf{Var}(s_i)/n \to 0$ as $n \to \infty$; and (c) $\max_{i,j}[\mathbf{Cov}(s_i, s_j)] \cdot \mathbf{Var}(s_i) \to 0$. These conditions can be achieved easily, *not only* from the random permutation. For example, one may generate i.i.d. random variables $s'_0, s'_1, \ldots, s'_{n-1}$, and then take $s_i = \sqrt{n}|s'_i|/\sqrt{\sum_{j=0}^{n-1}(s'_j)^2}, i = 0, 1, \ldots, n-1$. The following proposition specifies the limitation of $(1/n)\sum s_i$ when s'_i's are sampled from the normal distribution $\mathrm{N}(0, \sigma^2)$.

Proposition 3.3. *Consider a sequence $s = \{s_0, s_1, \ldots, s_{n-1}\}$. Assume s is uniformly distributed on the intersection of the sphere $\sqrt{n} \cdot S^{n-1}$ (i.e., the set $\{s \in \mathscr{R}^n : \sum_{i=0}^{n-1} s_i^2 = n\}$) and the positive cone $\{s \in \mathscr{R}^n : s_i \geqslant 0, 0 \leqslant i \leqslant n-1\}$. We have, as $n \to \infty$,*

$$\frac{1}{n}\sum_{i=0}^{n-1} s_i \to \sqrt{\frac{2}{\pi}} \quad a.s. \tag{3.7}$$

Note $\sqrt{2/\pi} \approx 0.7979$. This is consistent with our observations in simulations.

Proof. Let random variables $X_0, X_1, \ldots, X_{n-1}$ be i.i.d. standard normal ($\mathrm{N}(0,1)$) distributed. Denote the sequence as $X = \{X_0, X_1, \ldots, X_{n-1}\}$. From a property of the multivariate normal distribution, if we assign $s_i = \sqrt{n}|X_i|/\sqrt{\|X\|_2^2}$, where $\|X\|_2^2 = \sum_{i=0}^{n-1} X_i^2$, s will be uniform on $\sqrt{n} \cdot S^{n-1} \cap \{s : s_i \geqslant 0\}$, and vice versa. Hence,

$$\frac{1}{n}\sum_{i=0}^{n-1} s_i = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1} \frac{|X_i|}{\sqrt{\|X\|_2^2}} = \frac{\sum_{i=0}^{n-1}|X_i|/n}{\sqrt{\|X\|_2^2/n}}. \tag{3.8}$$

From the law of large number, we have

$$\frac{\sum_{i=0}^{n-1}|X_i|}{n} \to \mathbf{E}|X_1| \quad \text{a.s.} \tag{3.9}$$

Recall $X_1 \sim \mathrm{N}(0,1)$, it is not hard to verify that $\mathbf{E}|X_1| = \sqrt{2/\pi}$. Meanwhile, we have

$$\frac{\|X\|_2^2}{n} \to \mathbf{E}(X_1^2) \quad \text{a.s.} \tag{3.10}$$

Combining (3.8), (3.9), (3.10), and $\mathbf{E}(X_1^2) = 1$, we have the result in (3.7). □

We single out the above, because such an approach has been used in the literature to generate random magnitudes. Interestingly, a particular constant $\sqrt{2/\pi}$ can be computed. Readers can find that the above result holds for many other distributions, with possibly different constants.

For non-DC components in the inverse Fourier transform, we have the following result.

Proposition 3.4. *For $\varepsilon_{i,n}, i = 0, 1, \ldots, n-1$, satisfying $\sum_{i=0}^{n-1}\varepsilon_{i,n} = 0$ and $\sum_{i=0}^{n-1}\varepsilon_{i,n}^2 = n$, for s_i's that are defined in Proposition 3.3, we have*

$$\frac{1}{n}\sum_{i=0}^{n-1} \varepsilon_{i,n} s_i \to 0, \quad \textit{in probability}, \quad \textit{as } n \to \infty. \tag{3.11}$$

Proof. Given $s_i > 0$, the set of variables $\{s_j,\ j \neq i,\ 0 \leqslant j \leqslant n-1\}$ is uniform on the sector $\{s_j : j \neq i, s_j \geqslant 0, \sum_{j \neq i} s_j^2 = n - s_i^2\}$. It is not hard to see that random variable s_i and random variable $\sum_{j \neq i} s_j | s_i$ are negatively correlated. More specifically, as s_i increases, $\mathbf{E}(\sum_{j \neq i} s_j | s_i)$ decreases because the radius of the sphere becomes smaller. Hence,

$$\mathbf{E}\left(s_i \cdot \sum_{j \neq i} s_j\right) \leqslant (\mathbf{E}s_i)\left(\mathbf{E}\sum_{j \neq i} s_j\right).$$

Moreover, we have

$$\mathbf{E}\left[s_i\left(\sum_{j\neq i} s_j - s_i\right)\right] \leqslant \mathbf{E}s_i\left(\mathbf{E}\sum_{j\neq i} s_j - \mathbf{E}s_i\right) = (n-2)(\mathbf{E}s_i)^2.$$

Therefore, for $i \neq j$, we have

$$(n-1)\mathbf{E}(s_i s_j) = \sum_{j\neq i} \mathbf{E}(s_i s_j) \leqslant \mathbf{E}s_i^2 + (n-2)(\mathbf{E}s_i)^2.$$

The above is equivalent to

$$\mathbf{E}(s_i s_j) \leqslant \frac{1}{n-1}\mathbf{E}(s_i^2) + \left(1 - \frac{1}{n-1}\right)(\mathbf{E}s_i)^2.$$

Therefore,

$$\begin{aligned}\mathbf{Cov}(s_i, s_j) &= \mathbf{E}(s_i s_j) - [\mathbf{E}(s_i)]^2 \\ &\leqslant \frac{1}{n-1}\mathbf{E}(s_i^2) - \frac{1}{n-1}(\mathbf{E}s_i)^2 \\ &= \frac{1}{n-1}\mathbf{Var}(s_i). \end{aligned} \tag{3.12}$$

Readers may compare the above with inequality (3.6) in the proof of Theorem 3.1. Recall $\mathbf{Var}(s_i) \leqslant \mathbf{E}(s_i^2) = 1$, the rest of the proof is nearly identical with the proof of Theorem 3.1. We have

$$\begin{aligned}\mathbf{Var}\left(\frac{1}{n}\sum \varepsilon_{i,n} s_i\right) &< \frac{\mathbf{Var}(s_i)}{n^2}\left[n + \frac{n^2}{n-1}\right] \\ &< \frac{3}{n}\mathbf{Var}(s_i) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

It is easy to verify $\mathbf{E}((1/n)\sum \varepsilon_{i,n} s_i) = 0$. We prove the theorem. □

So far, we have considered the cases when the magnitudes are random. Now we consider random phases. Let $\mathbf{C}^1$ denote the unit circle on the complex plain: $\mathbf{C}^1 = \{x | x \in \mathbb{C}, \|x\|_2 = 1\}$. We have the following theorem.

Theorem 3.5. *Given $\varepsilon_{i,n}, i = 0, 1, \ldots, n-1$, satisfying $\sum_{i=0}^{n-1} \varepsilon_{i,n}^2 = n$, for random i.i.d. signals $s_i \sim$ Uniform $(\mathbf{C}^1)$, we have*

$$\mathbf{E}\left(\sum_{i=0}^{n-1} \varepsilon_{i,n} s_i\right) = 0 \quad \textit{and} \quad \mathbf{Var}\left(\sum_{i=0}^{n-1} \varepsilon_{i,n} s_i\right) = n.$$

It only takes standard statistical calculation to verify the above. We omit the details. This theorem shows that there is no dominating component in the inverse Fourier transform of random phases. In fact, all the components have the same variance. Hence, a reconstruction with randomly distorted phases is *unlikely* to be close to the original.

4. Discussion

Related works: A number of researchers have given their own ways to interpret the relative importance of phases. For example, Goodman illustrates it from a simple example (Goodman, 1984). Pearlman and Gray (1978) show that for the same amount of distortion, phases must be encoded with 1.37 more bits than magnitudes. Srinivasan and Chandrasekaran (1966), in their experiment of atomic structure reconstruction, show that the phase-only reconstruction preserves much of the correlation between signals. In this paper, we give a justification different from all of the above.

Issues on computing: Regarding to computing, which is another important aspect of the reconstruction problem, people have found that numerical reconstruction from Fourier phases is much easier than from Fourier magnitudes. In fact, based on phase-only information, the reconstruction is equivalent to solving a system of linear equations. Ma (1991) gives a condition on the uniqueness of the reconstruction and this is a problem of linear programming. Meanwhile, Ma (1991) also mentions that reconstructions based on magnitude-only information are ill-posed and the solutions have strong combinatoric flavor. Actually, reconstruction from magnitudes is a blind deconvolution problem, and has been extensively studied. Readers can refer to Stockman et al. (1975) and Stark (1987).

5. Conclusion

From a statistical perspective, we give an interpretation of the importance of phase information in signal/image reconstructions. We illustrate that a random distortion of the phases can dramatically distort the reconstructed signal, while a random distortion of the magnitudes will not, because the convolution sequence in the latter case is, with a large probability, close to a Dirac sequence.

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