Fundamental Groups of low-genus Lefschetz Fibrations

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AMS Special Session on Topological Interactions of Contact and Symplectic Manifolds

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Why Lefschetz Fibrations?

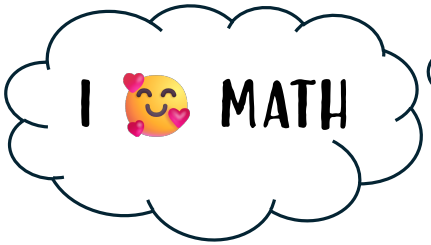
Image from breakthroughprize.org

symplectic manifolds have the structure of a Lefschetz fibration

Lefschetz fibrations in symplectic geometry, 1998

Lefschetz pencils have symplectic structures

The topology of symplectic manifolds, 2001



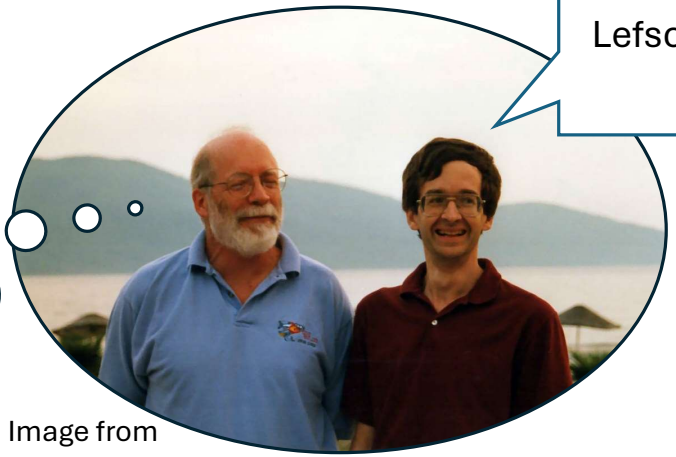
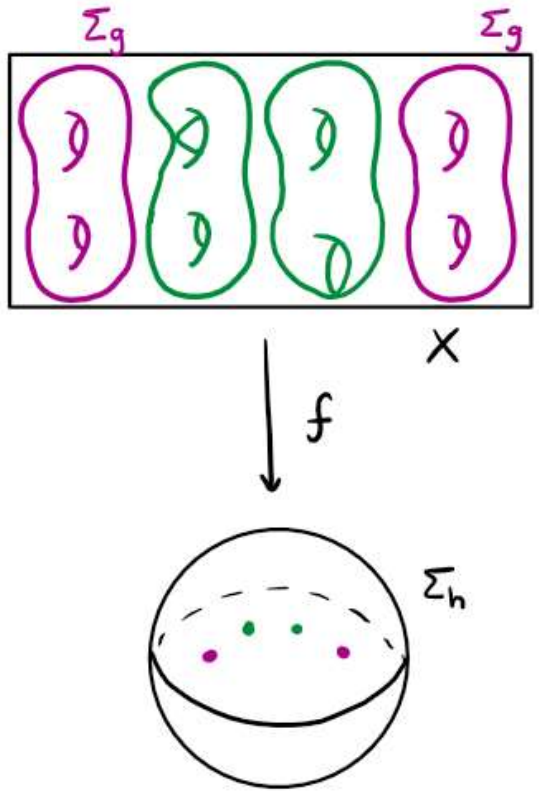


Image from celebratio.org

Part I

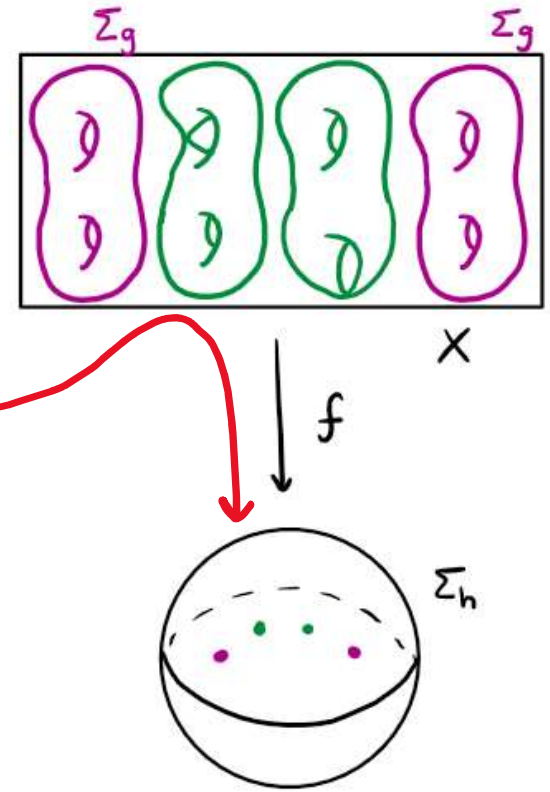
Definitions

Definition: Lefschetz fibration



Definition: Lefschetz fibration

1 $f: X^4 \to \Sigma_h$ is a smooth surjection



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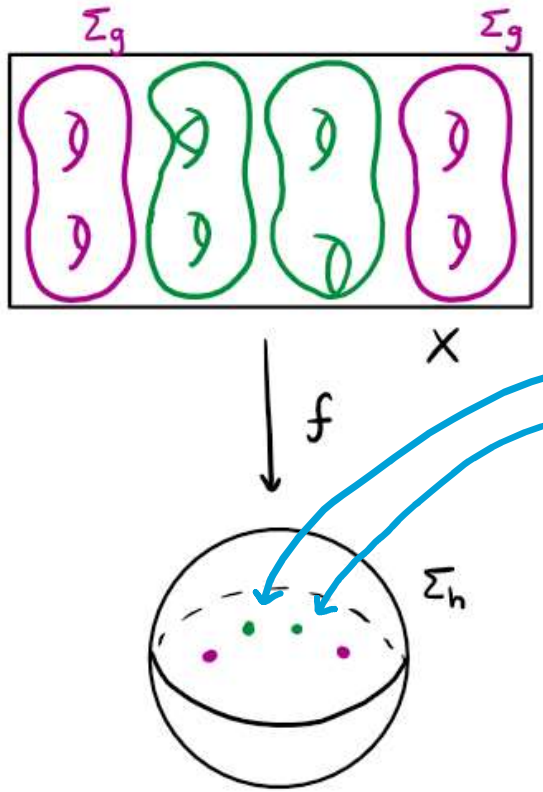

2 Finitely many critical values $q_1, \dots, q_n$

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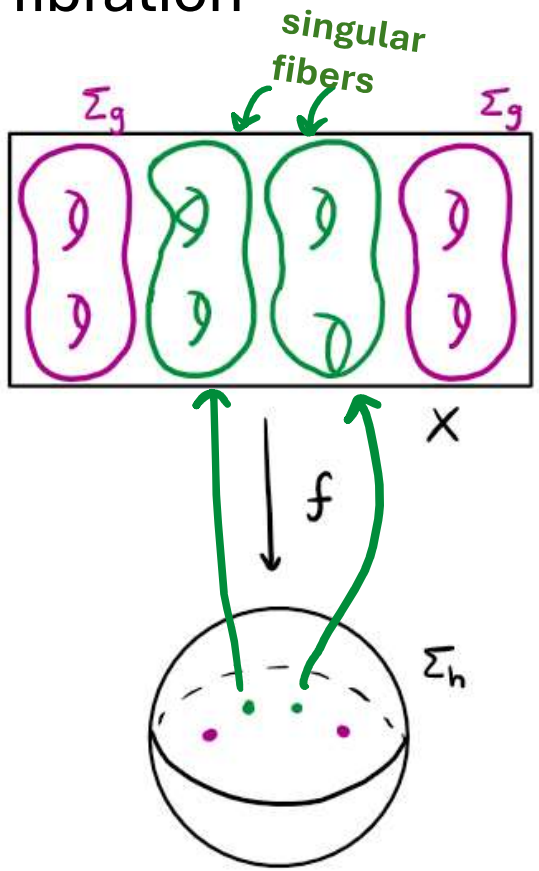

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3 each $f^{-1}(q_i) \in X$ has a local coord chart in which $f(z,w) = zw$

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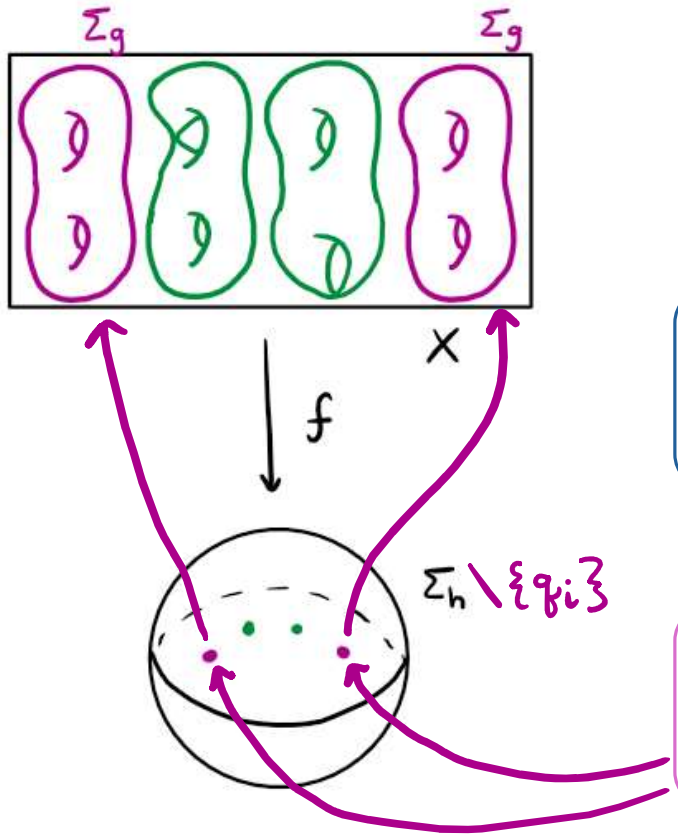
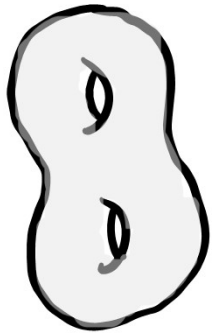
4 $f^{-1}(b) =$ regular fiber (genus-g surface)



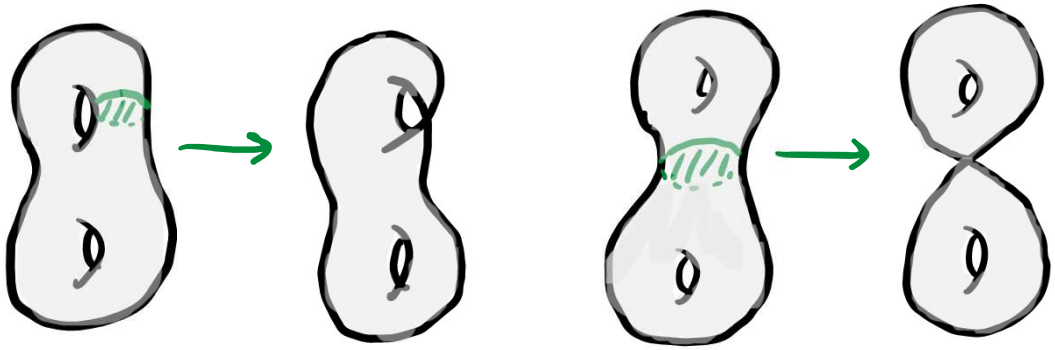
Definition: Lefschetz fibration

Regular fiber



genus 2

Singular fibers



non-separating

separating

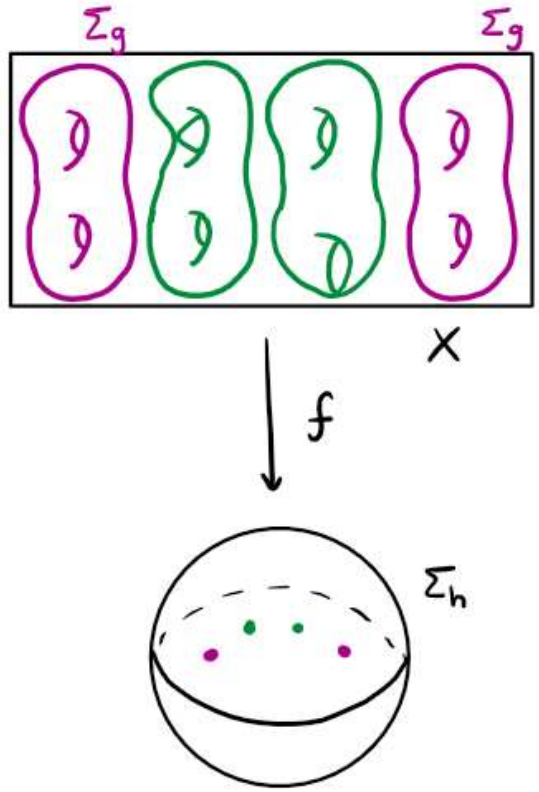
Defn: = vanishing cycle

Definition: Lefschetz fibration

Remarks:

1. the genus of the Lefschetz fibration is the genus of a **regular fiber**

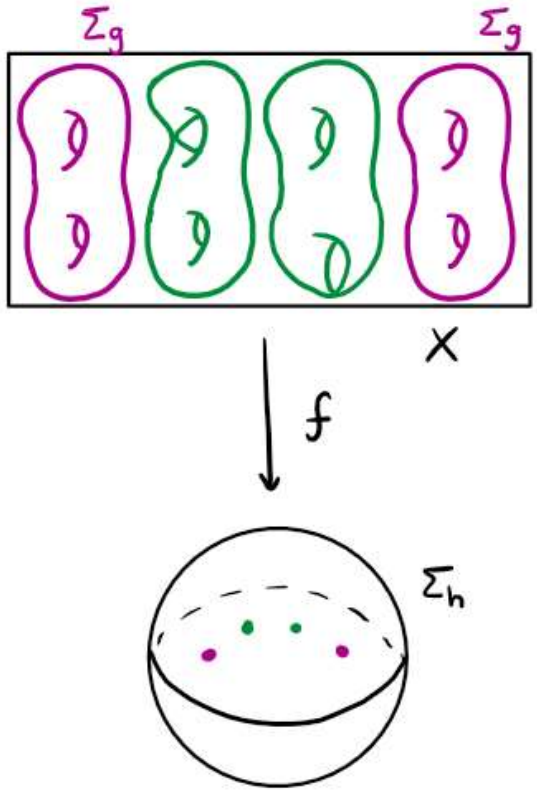
2. the **monodromy** determines the Lefschetz fibration



Definition: monodromy of a Lefschetz fibration

Definition:

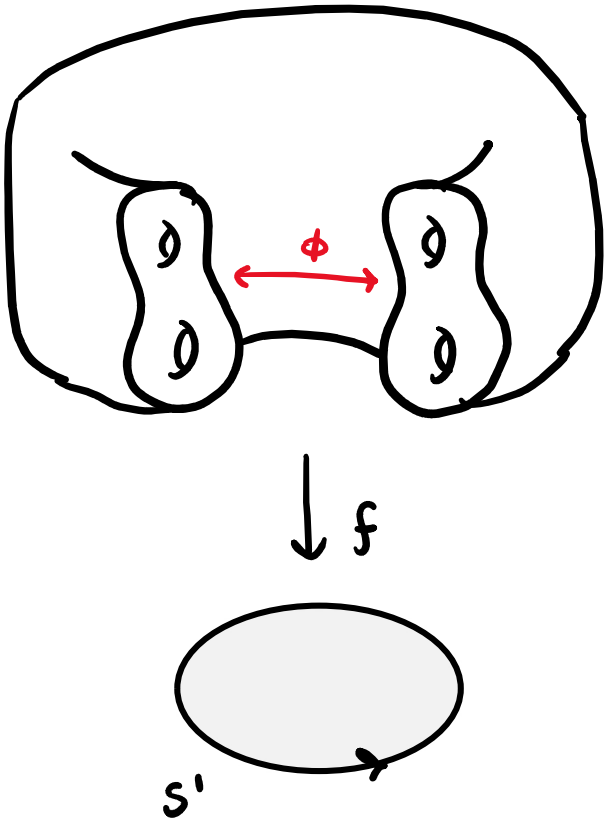
- embedded S^1 in base space
- pre-image is $S^1 \times \Sigma_g$
- the **monodromy** is the self-diffeo of a regular fiber Σ_g to itself



Definition: monodromy of a Lefschetz fibration

Remarks:

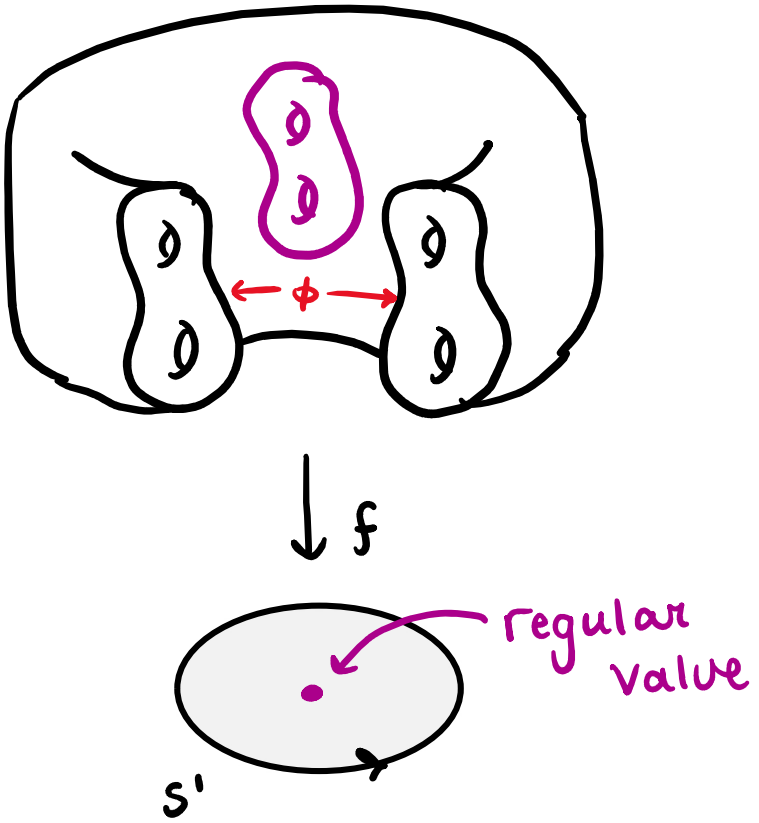
- the **<u>monodromy</u>** is the self-diffeo of a regular fiber Σ_g to itself
- Denoted ϕ
- $\phi \in Mod(\Sigma_g)$



Definition: monodromy of a Lefschetz fibration

No critical values in D^2

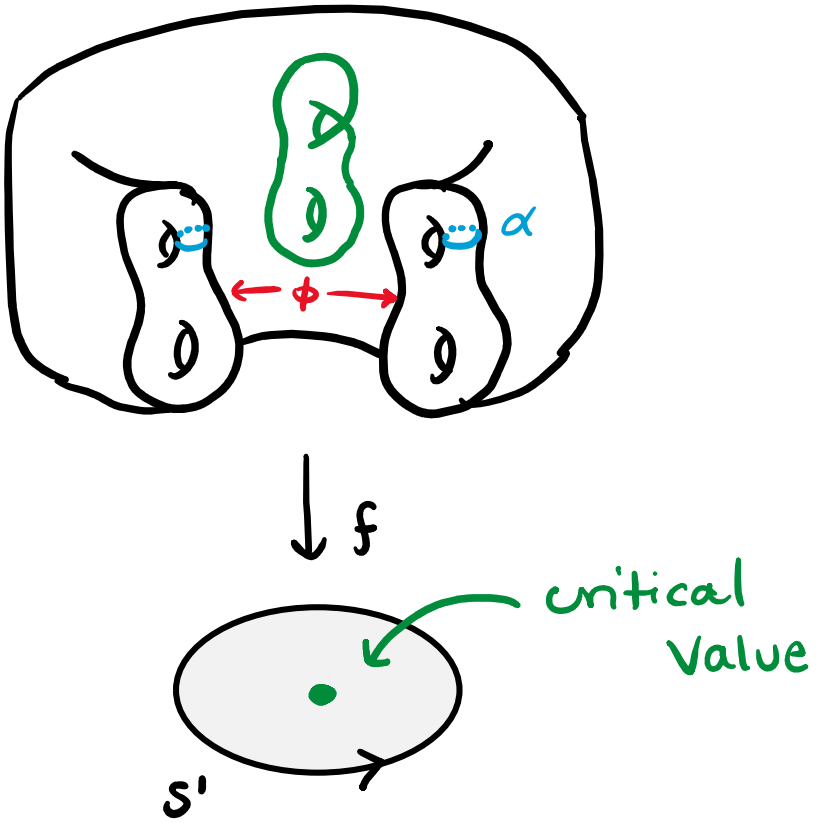
- $\phi =$ how to glue Σ_g to itself
- $\phi = Id$



Definition: monodromy of a Lefschetz fibration

One critical value in D^2:

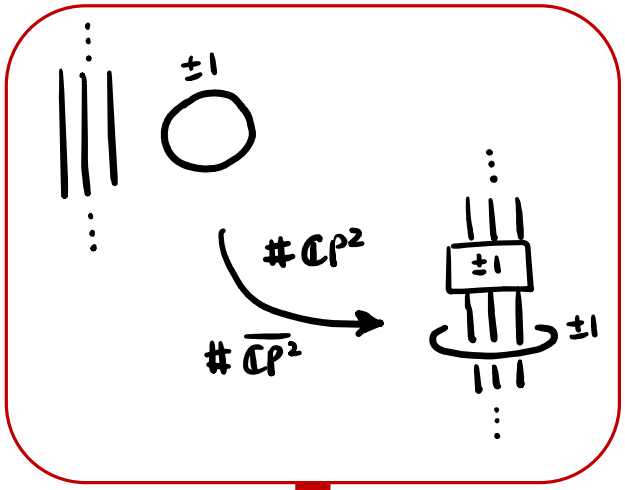
- $\phi = \tau_\alpha$
- Left-handed Dehn twist about vanishing cycle α



Part II

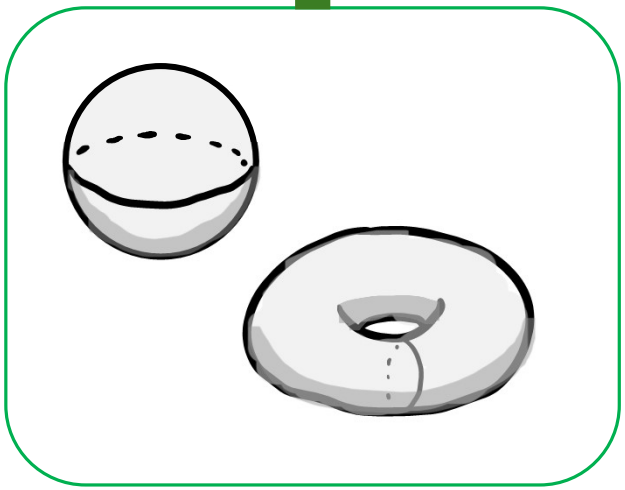
Motivation

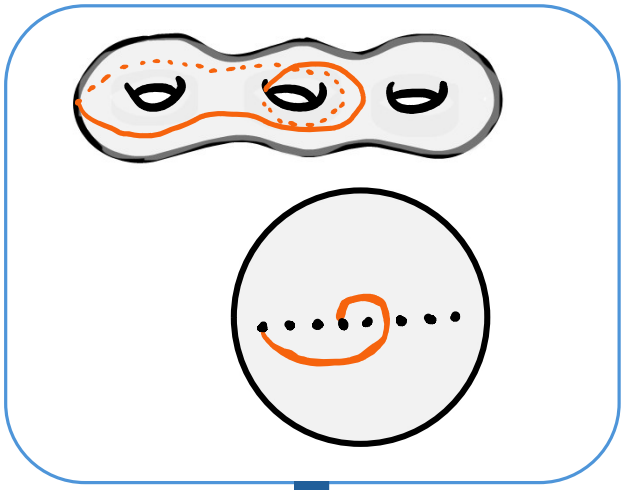
Asking Q's about the fundamental group



blowing up does not change π_1 of manifold

genera-0 and 1 already classified





Q's about curves on surfaces (or braids on n-strands)

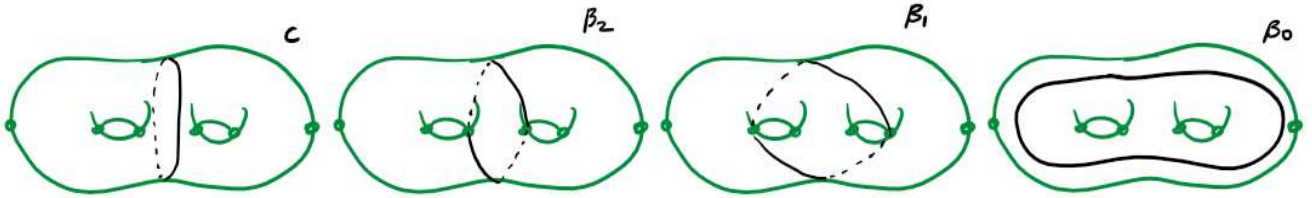
What's known:

Gompf: every finitely presented group is π_1 of some closed symplectic 4 manifold

A new construction of symplectic manifolds, 1995

Image from Mustafa Korkmaz's website

given $\Gamma = \langle$ generators | relations $\rangle$ & Matsumoto's relation:

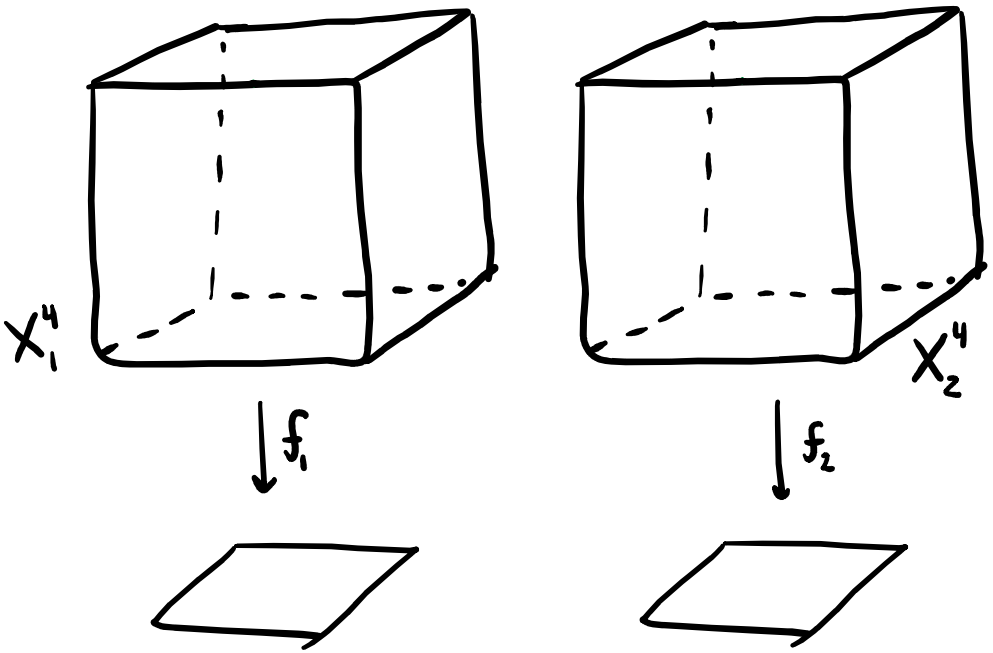


...I can give you the Lefschetz fibration X s.t. $\pi_1(X) = \Gamma$

Lefschetz Fibrations and an invariant of finitely presented groups, 2009

Korkmaz needs fiber sum:

Question: How do I glue 2 Lefschetz Fibrations together?



Answer:

Fiber Sum $X_1 \#_f X_2$

1. Let $C_1 = F \times D^2$ in X_1
2. And $C_2 = F \times D^2$ in X_2

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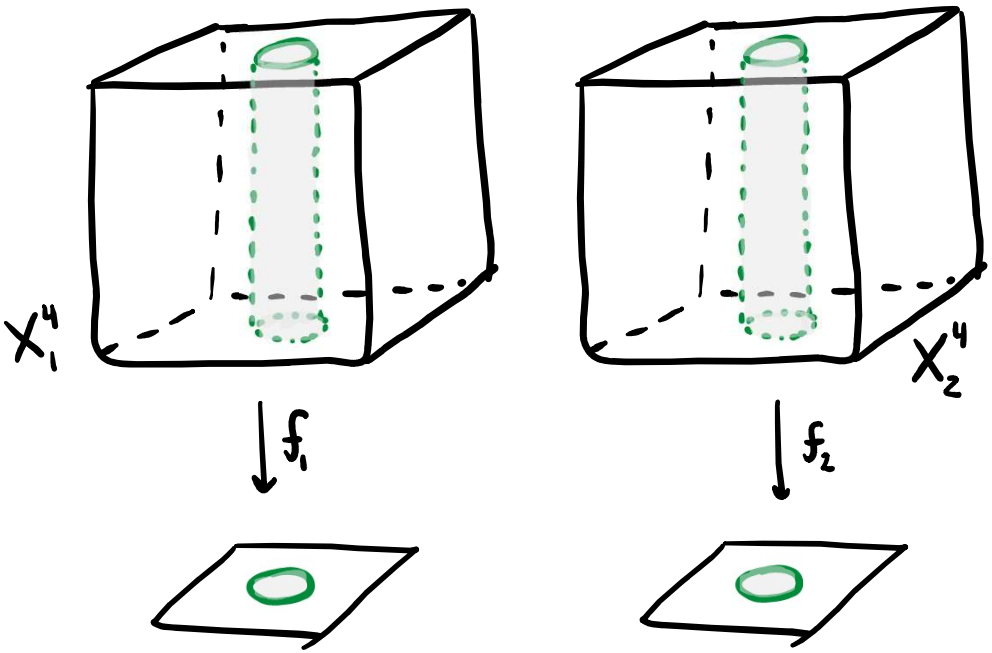

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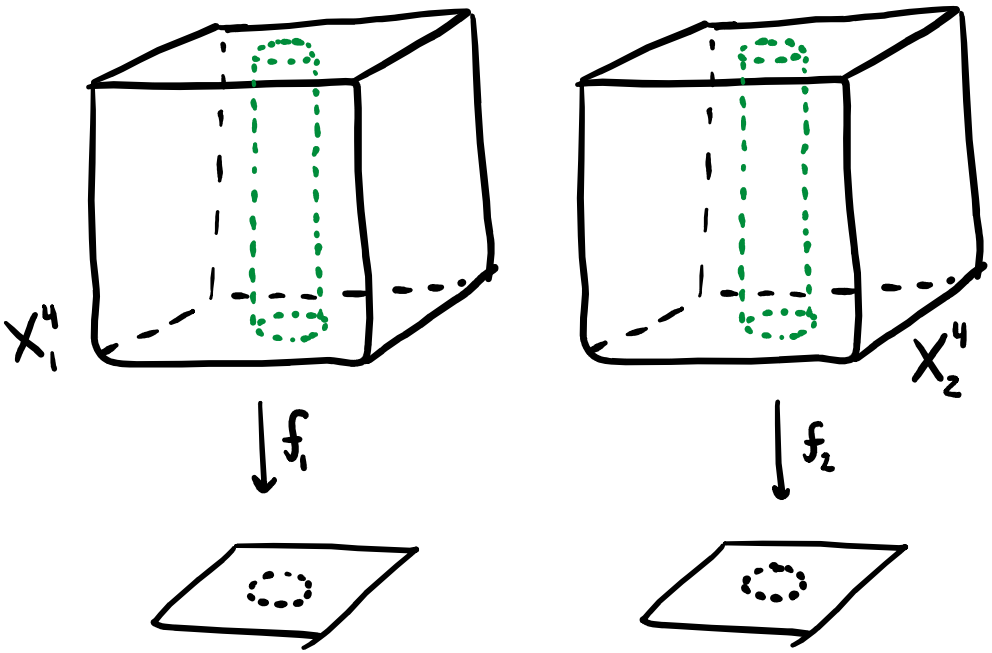

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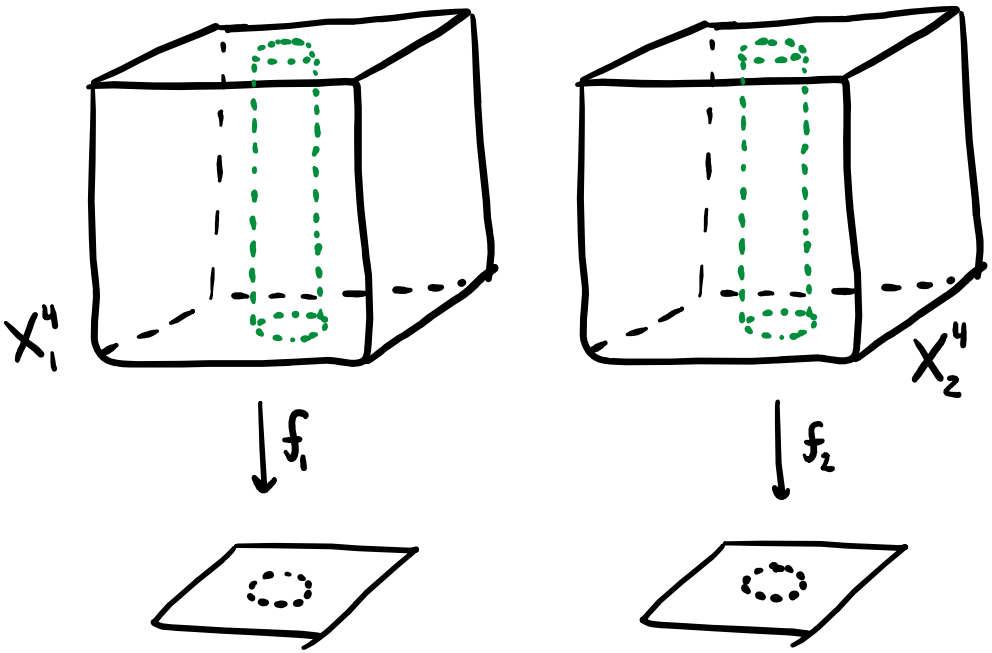


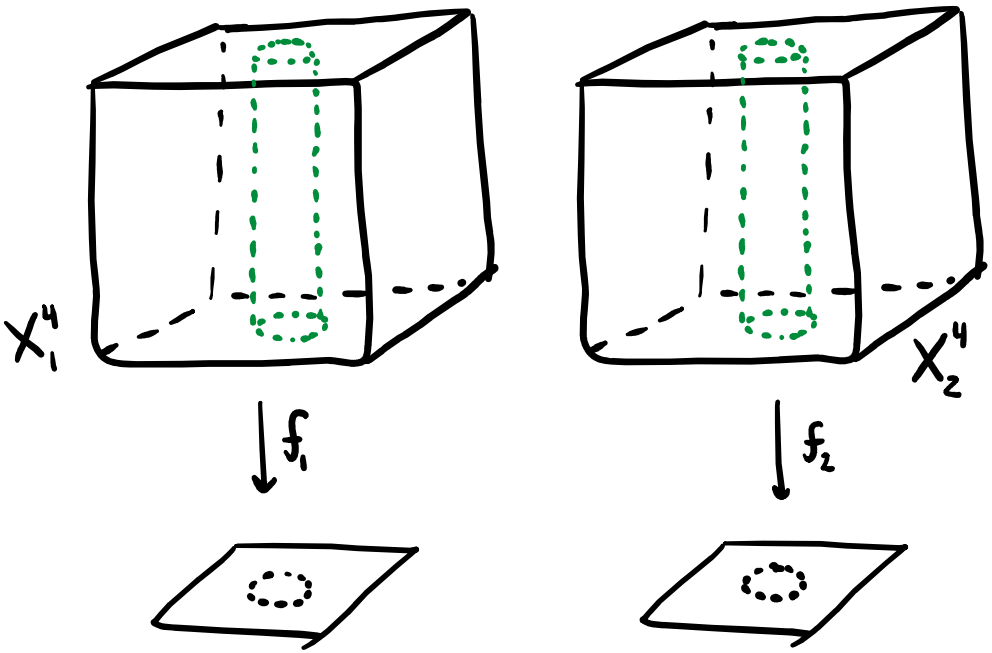
Answer:

Fiber Sum $X_1 \#_f X_2$

1. Let $C_1 = F \times D^2$ in X_1
2. And $C_2 = F \times D^2$ in X_2
3. Remove C_i from X_i
4. Glue $\partial(X_1 - C_1)$ to $\partial(X_2 - C_2)$ by a fiber-preserving, orientation reversing diffeo

Korkmaz needs fiber sum:

Question: How do I glue 2 Lefschetz Fibrations together?



Answer:

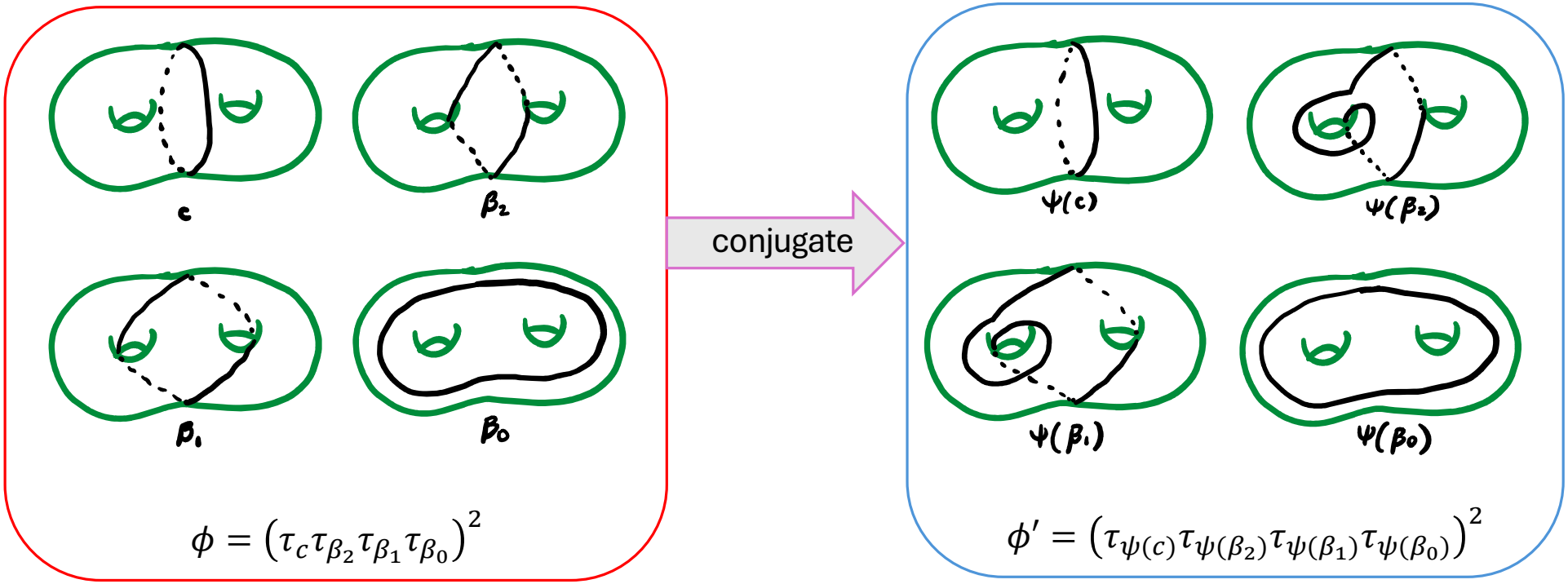
Fiber Sum $X_1 \#_f X_2$

Remark:

New monodromy is

$$\phi_1 \phi_2 = Id$$

Korkmaz's construction:



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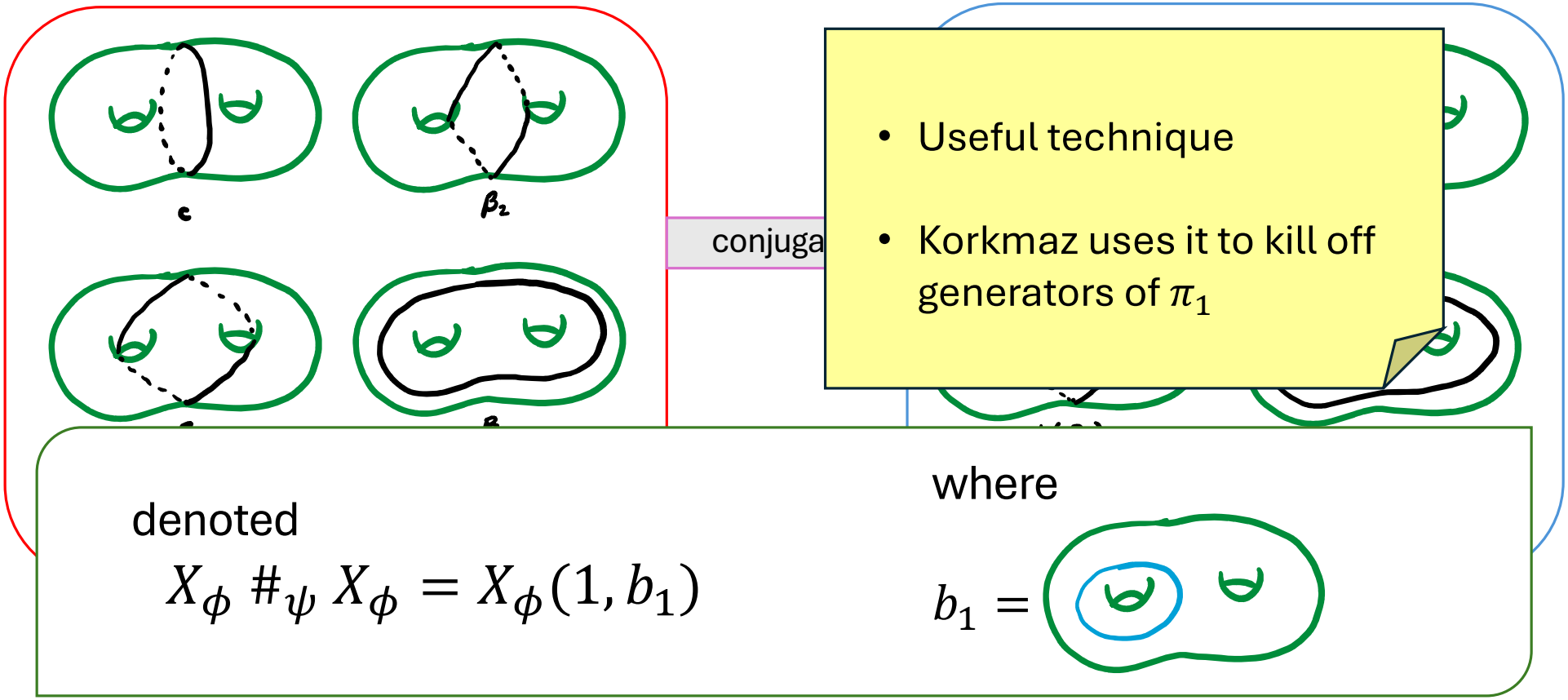
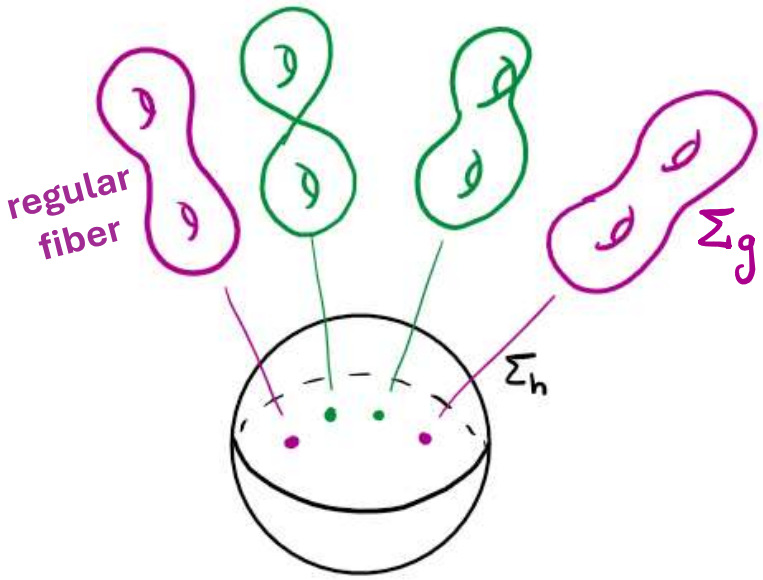


Part III

Results

Goal:

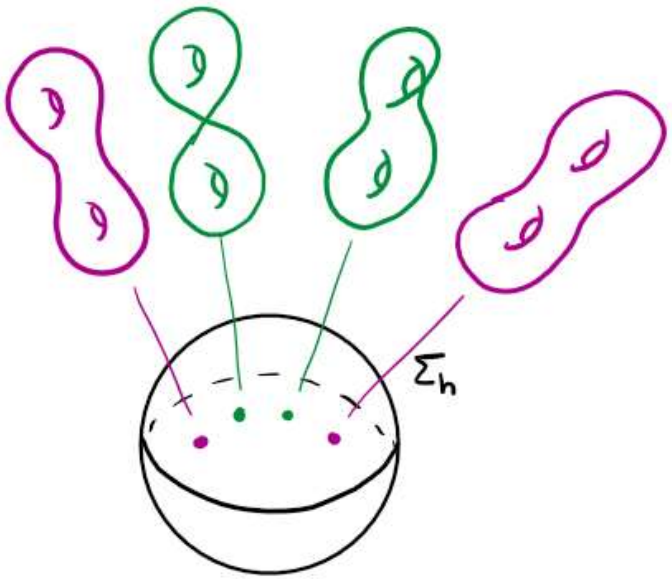


assume genus of base space is $h = 0$

fix **genus** of a Lefschetz fibration X

find all Γ such that $\pi_1(X) = \Gamma$

Results:



Possible π_1 for a genus-2 LF over S^2:

$$0 = \langle \ | \ \rangle$$

$$\mathbb{Z} = \langle a \mid \ \rangle$$

$$\mathbb{Z}_n = \langle a \mid a^n \rangle$$

$$\mathbb{Z} \oplus \mathbb{Z} = \langle a, b \mid [a, b] \rangle$$

$$\mathbb{Z}_n \oplus \mathbb{Z} = \langle a, b \mid [a, b], a^n \rangle$$

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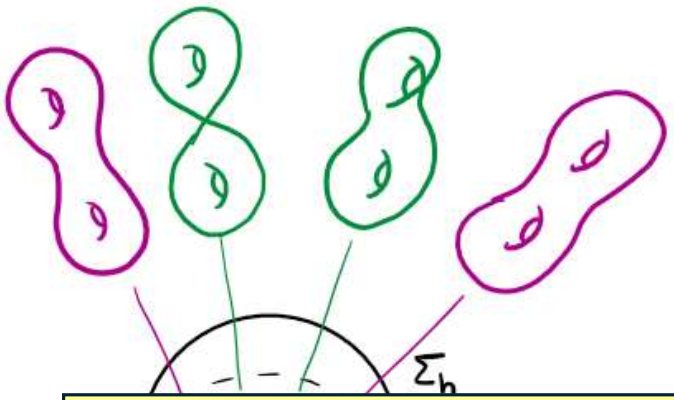

Future questions:

1. Always ≤ 2 generators?
2. Always Abelian?

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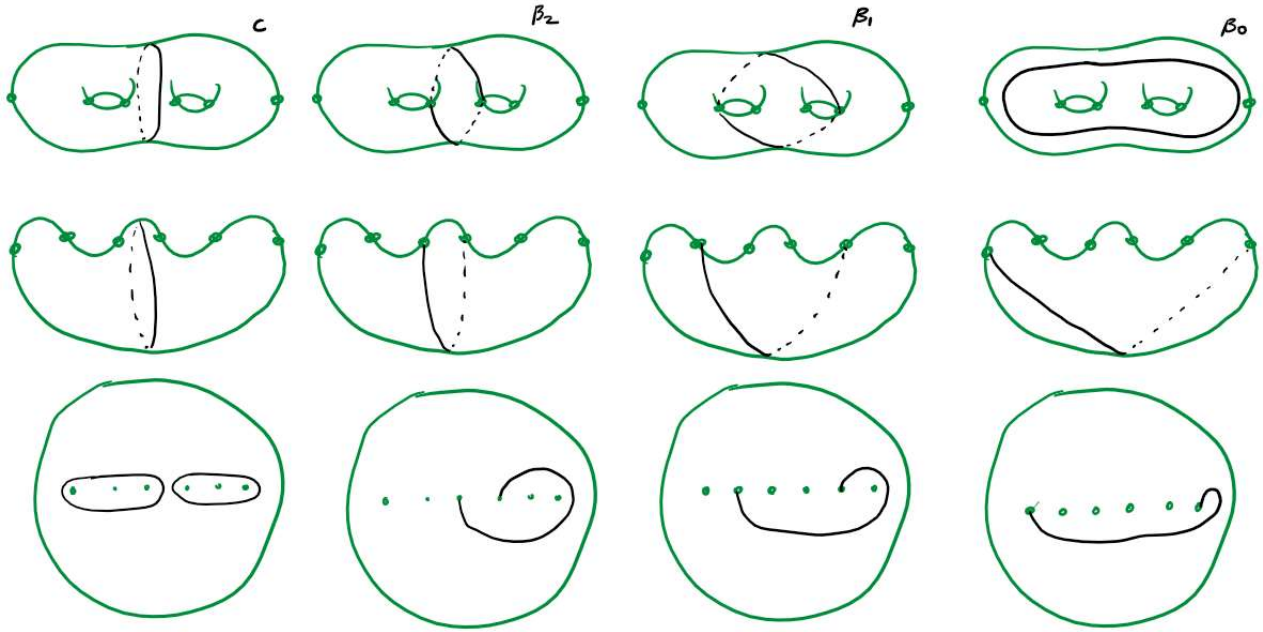
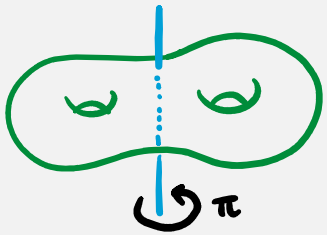
Current progress:

Hyperelliptic LF
&
Hyperelliptic involution

Arcs and embedded loops on S^2 with 6 marked points



If involutive with respect to 

, then Abelian with ≤ 2 generators

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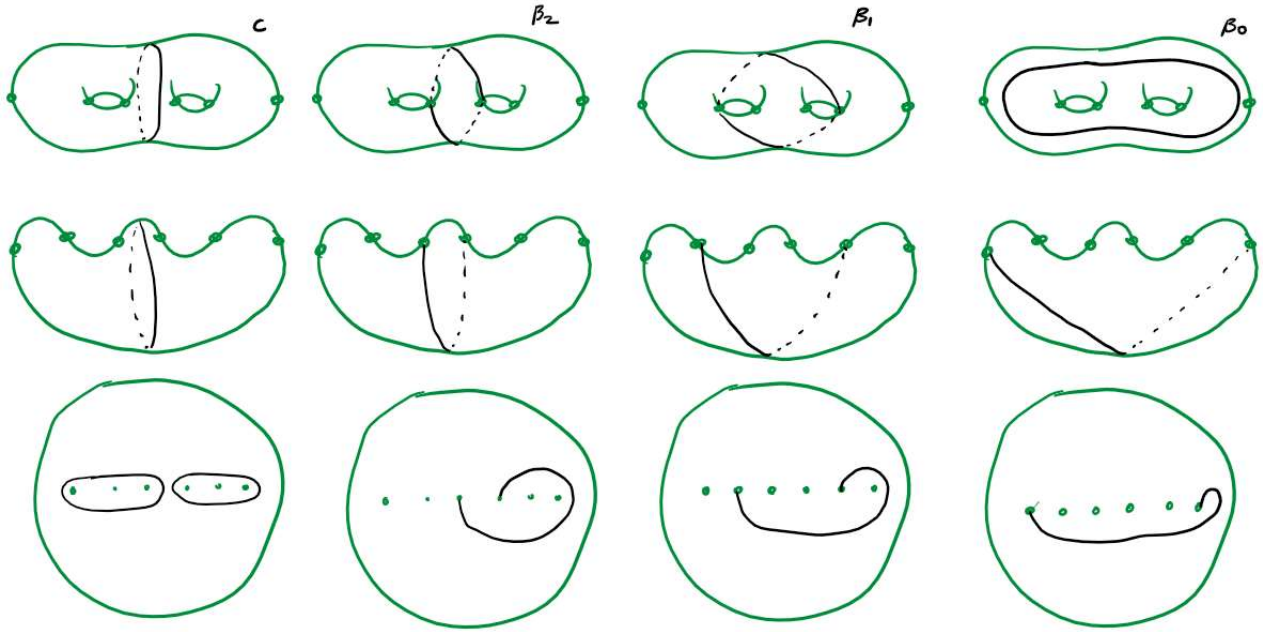
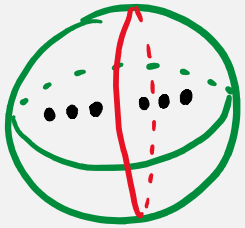


If involutive with respect to  , then Abelian with ≤ 2 generators

Thanks!

sknavel3@gatech.edu

Florida State University's Land Acknowledgement:

> Florida State University acknowledges that its Florida campuses are located on the ancestral and traditional homelands of the Seminole Tribe of Florida, the Miccosukee Tribe of Indians of Florida, the Apalachee, Seminole and Muscogee Nations, the ancient Calusa, Uzita and Tocobaga, and others. We pay respect to the resiliency of their tribal members, past and present, and to all Indigenous peoples.